62 + 6 Bilagor

DOKTORSAVHANDLINGAR
VID
CHALMERS TEKNISKA HÖGSKOLA
Nr. 62

A STUDY OF THE UNITARY REPRESENTATIONS OF SOME SPACE-TIME TRANSFORMATION GROUPS

BY

STAFFAN STRÖM

GÖTEBORG 1967

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Tekn. lic.

AKADEMISK AVHANDLING

SOM MED TILLSTÅND AV SEKTIONEN FÖR TEKNISK FYSIK VID CHALMERS TEKNISKA HÖGSKOLA FRAMLÄGGES TILL OFFENTLIG GRANSKNING FÖR TEKNOLOGIE DOKTORSGRADS VINNANDE TORSDAGEN DEN 25 MAJ 1967 KL. 10 Å PALMSTEDTSKA SALEN I ADMINISTRATIONSBYGGNADEN VID CHALMERS TEKNISKA HÖGSKOLA, SVEN HULTINS GATA, GÖTEBORG

GÖTEBORG
ELANDERS BOKTRYCKERI AKTIEBOLAG
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The present paper is an introduction to and a summary of a thesis comprising the following six papers:

I. Ström, S., On the matrix elements of a unitary representation of the homogeneous Lorentz group, Arkiv Fysik *29*, 467 (1965).

II. Ström, S., A note on the matrix elements of a unitary representation of the homogeneous Lorentz group, Arkiv Fysik *33*, 465 (1967).

III. Ström, S., On the contraction of representations of the Lorentz group to representations of the Euclidean group, Arkiv Fysik *30*, 267 (1965).

IV. Ström, S., Construction of representations of the inhomogeneous Lorentz group by means of contraction of the $(1+4)$ de Sitter group, Arkiv Fysik *30*, 455 (1965).

V. Kihlberg, A. and Ström, S., On the unitary irreducible representations of the $(1+4)$ de Sitter group, Arkiv Fysik *31*, 491 (1966).

VI. Ström, S., Some remarks on the decomposition of a unitary representation of the Lorentz group with respect to representations of non-compact subgroups, Arkiv Fysik *34*, 215 (1967).

Introduction

The usefulness of group theory for the treatment of a physical system which is characterized by a symmetry property is generally accepted. It is amply illustrated by the role played by the Poincaré group, the Lorentz group, the rotation group, the permutation groups, etc in various branches of contemporary theoretical physics. However, the usefulness of group theory is not confined to the analysis of symmetries. The development of the theory of group representations, in particular the study of the unitary irreducible representations (abbreviated UIR's in the following), includes, among other things, a systematic extension of the idea of Fourier series in as much as the UIR's define an (in some generalized sense) orthogonal and complete system of functions on the group. The set of functions obtained in this way usually provides a practical and transparent scheme for detailed calculations irrespective of whether a symmetry is present or not.

In the study of relativistic quantum mechanical kinematics the natural framework is provided by the representation theory for the Poincaré group. In a fundamental paper [1] E. P. Wigner determined the UIR's of the Poincaré group. A detailed and explicit knowledge of various aspects of these representations is required e.g. in the construction of relativistic free-particle states and in the study of relativistic many-particle reactions. A realization of the one-particle states was given in a subsequent paper by V. Bargmann and E. P. Wigner [2]. H. Joos gave the reduction of a direct product of two UIR's of the Poincaré group into irreducible components in reference [3], a paper which contributed greatly to the renewed interest in the proper formulation of the relativistic quantum mechanical kinematics. This reduction is of basic importance for the treatment of relativistic many-particle reactions. An excellent exposition of the whole field and a comprehensive list of references is given in [4].

One of the main problems of a relativistic theory of reactions consists in the construction of convenient bases in the space of state vectors. Group-theoretically this amounts to choosing suitable realizations of the relevant UIR's of the Poincaré group. For the treatment of relativistic scattering experiments involving polarized particles the so-called helicity states [4] are often the most useful ones. The helicity of a particle is defined to be the component of the total angular momentum along its direction of motion. This quantity is invariant under the three-dimensional Euclidean group $E(3)$. An alternative to the helicity formalism is therefore obtained by using an $E(3)$-basis for the UIR's of the Poincaré group [5] — [8]. There are many situations in which this $E(3)$-basis is very suitable. We refer to [9] for a number of explicit examples.

A different classification of the states is exploited in the formalism in which the wave function of a free particle is described in terms of UIR's of the Lorentz group. The basic connection, the decomposition of a UIR of the Poincaré group into UIR's of the Lorentz group was first given by Chou Kuang-Chao and L. G. Zastavenko [10] and later independently by H. Joos [3]. Other authors have extended this formalism to two-particle states and applied it to two-particle scattering [11], [12]. The connections between the above-mentioned formalisms are given by the transformations between the corresponding bases of the UIR's of the Poincaré group [3], [13].

Another aspect of the quantum mechanical relativistic kinematics which has recently received much attention is the crossed-channel partial wave analysis where the total momentum is a spacelike vector [14] — [19]. The relevant UIR's of the Poincaré group are then partly characterized by the UIR's of the three-dimensional Lorentz group. These latter representations are labelled by a number analogous to the angular momentum quantum number, which, however, is now complex.

The non-relativistic coordinate transformations are usually called Galilei transformations. They can be obtained from the Poincaré transformations in the limit of infinite velocity of light. Intuitively, therefore, the group of Galilei transformations is in some sense a limiting case of the Poincaré group. E. Inönü and E. P. Wigner have introduced a mathematical concept, contraction of a Lie group, which is suitable for describing situations of this kind [20]. In short it can be described as follows: a Lie group G_2 can be obtained from another Lie group G_1 by contraction if there exists a non-singular similarity transformation of G_1 which depends on a parameter λ and which in the limit $\lambda \to 0$ becomes singular and if in this limit the group G_2 is obtained as a result of the transformation.

In physical applications it is the UIR's of the group involved that are of greatest importance. One may therefore ask whether the contraction is a useful concept also in discussing the UIR's of the groups. It turns out that in order to obtain a faithful representation of G_2 one must combine the prescription $\lambda \to 0$ with some other limiting procedure. The one which has turned out to be most useful for groups of physical importance is one in which limits of certain sequences of UIR's of G_1 are considered [20]. However, there does not exist a general theory for contraction of representations and so far only a few cases have been studied.

The Poincaré group itself can be obtained by contraction, namely by contraction of the $(1+4)$ or $(2+3)$ de Sitter group, in the following denoted $L(1+4)$ and $L(2, 3)$ respectively. This fact raises the question whether it is possible to introduce $L(1, 4)$ or $L(2, 3)$ as a fundamental space-time transformation group, replacing the Poincaré group. One important aspect of this question is the possibility that by contraction some important restrictions may be lost. Thus for example, one knows that when passing from Poincaré invariance to Galilei invariance, the TCP theorem and the connection between spin and statistics are lost, properties which are of importance also in non-relativistic situations. Correspondingly the difference between Poincaré and de Sitter invariance may be of importance in situations apparently not affected by the limit involved in the contraction. The problems which have been considered in this context include: the formulation of a de Sitter covariant field theory [21], [22] and the possibility of constructing localized states in de Sitter space [23]. However, the connection between the UIR's of the de Sitter groups and those of the Poincaré group has received little attention (see, however, [24]).

Another field of theoretical physics which has been very actively pursued in the last few years is that of higher symmetries of elementary particle interactions. Under this heading we also include the attempts of combining space-time and internal symmetries. In contrast to the above-mentioned kinematical formalisms which are independent of the model of interactions which is used, one is here primarily interested in the consequences of specific assumptions concerning the interaction. The theories are usually Lagrangian theories which are generalizations in various directions of the original Dirac theory for the electron. For an introduction to this field we choose to give only two references, [25] and [26], where, however, the reader will find extensive lists of further references.

In this field many groups which are closely related to the Poincaré group and the Lorentz group appear. Among other things it has been suggested that as a relativistic generalization of the $SU(6)$ symmetry one should consider a group which is a semidirect product of the Poincaré group and an internal symmetry group chosen as $SL(6, C)$ [26]. $SU(n)$ and $SL(n, C)$ denote the group of nxn unitary complex matrices and the group of nxn complex matrices of determinant one respectively. Since it is extremely cumbersome to derive explicit results concerning e.g. the S-matrix in such a model, one has instead considered a simpler model where $SL(6, C)$ is replaced by $SL(2, C)$ [27], which is homo-morphic to the Lorentz group.

The internal symmetry group may be chosen in different ways but a common feature is that it must contain $SL(2, C)$ as a subgroup. In such a theory the elementary particles are classified as members of a basis of an infinite-dimensional unitary representation of the internal symmetry group. Another development of this field is the attempts to construct infinite-component wave equations with realistic mass spectra (see e.g. [28]). Also $L(1, 4)$ appears here in new contexts. There are UIR's of $L(1, 4)$ which can describe all the bound states of the non-relativistic hydrogen atom. The terms "spectrum-generating group" and "dynamical group" have been introduced for groups having this property with respect to the bound states of an arbitrary system [29] — [32]. $L(1, 4)$ here serves as a test case for future generalizations.

Summary of the papers I-VI

It is with the developments described above as a background that the present thesis should be viewed. In the thesis a number of physically important properties of the UIR's of various space-time transformation groups are studied. We use mainly infinitesimal methods, which are most familiar to physicists and which involve quantities with a direct physical interpretation, but we also point out connections with global treatments whenever suitable and instructive.

Within the formalism proposed and developed in [10], [11], [12] the explicit determination of the "relativistic spherical functions", i.e. bases for the UIR's of the Lorentz group defined on the parameter space of the group, and the matrix elements of the operator corresponding to a finite Lorentz transformation, becomes of interest. They are also important for the calculation of e.g. decay widths in models where the Lorentz group or $SL(2, C)$ appears as an internal symmetry group [33]. In *Paper I* we present a calculation of the matrix elements based on a systematic use of the infinitesimal relations. It turns out that in the general case the matrix elements, which are computed in an angular momentum basis, satisfy a fourth order differential equation, whereas in special cases they satisfy a second order differential equation, the solution of which may be expressed in terms of hypergeometric polynomials. In particular, this is the case for a set of matrix elements which form a basis for a UIR of the Lorentz group. We have here arrived at a new simple analytic form of the relativistic spherical functions of an arbitrary UIR of the Lorentz group belonging to the main series. This is an example of a case where the infinitesimal method is the most convenient one for obtaining a specific result.

In [34] a realization of the UIR's of the Lorentz group and the explicit form of a angular momentum basis is given. From these results an integral representation for the matrix elements studied in paper I can be deduced. *Paper II* is devoted to an investigation of this integral representation. The main result is a new expression for the general matrix element as an explicit sum involving hypergeometric and generalized hypergeometric functions.

In the treatment of non-relativistic problems the full Galilei invariance is seldom exploited, only invariance under $E(3)$ is actually used. In this way the spherical functions associated with $E(3)$ become of interest. In *Paper III* it is shown how practically all aspects, global and infinitesimal, of the representation theory of $E(3)$ can in fact be obtained by contraction of appropriate relations for the Lorentz group. We also point out the connection between the multiplier representations of the Lorentz group as given in [34] and the representations of $E(3)$ obtained by induction (see e.g. [35]).

In the contraction of $L(1, 4)$ to the Poincaré group the four-dimensional rotation subgroup of $L(1, 4)$ goes over into $E(3)$. One might therefore ask whether it is possible to obtain in this way an $E(3)$ basis realization of the UIR's of the Poincaré group. The problem is complicated by the fact that the basis is labelled by a continuous variable, the energy. In *Paper IV* we show that it is possible to use contraction of the representations also in this more complicated situation. Our derivation involves two steps. In the first step we make a change of basis in the representation space which is accomplished by means of some new relations for the Clebsch — Gordan coefficients. In the second step we specify which particular sequences of UIR's of $L(1, 4)$ that shall be considered in order to obtain as a result of the contraction the UIR's of the Poincaré group. We show that in this way we can obtain all the UIR's of the Poincaré group in infinitesimal form in an $E(3)$-basis.

We have above discussed a number of possible physical applications of the UIR's of $L(1, 4)$. The classification of these representations has been treated with infinitesimal methods in [36], [37], [38]. These results were used in paper IV. A. Kihlberg [39] has given a method for the explicit construction of the UIR's of a class of pseudo-orthogonal groups which includes $L(1, 4)$. In *Paper V* this method is used to construct the UIR's of $L(1, 4)$. From the point of view of the method of [39] this case is of particular interest. It is the simplest case in which one redundant variable appears in the carrier space chosen for the representations. We show that the complications which arise are not serious and the construction can be carried out in a simple and explicit manner to give all the UIR's of $L(1, 4)$.

In the formalism which uses explicitly the UIR's of the Lorentz group the problem of decomposing the UIR's of the (four-dimensional) Lorentz group with respect to the UIR's of the three-dimensional Lorentz group arises in the crossed-channel partial wave analysis. The same decomposition is important also for the discussion of the limit of vanishing total (space-like) momentum within the canonical little group formalism for the UIR's of the Poincaré group. In *Paper VI* we study this decomposition by means of a new infinitesimal method. We also solve the problem of decomposing a UIR of the Lorentz group with respect to the UIR's of the two-dimensional Euclidean group.

Acknowledgements

On this occasion it is a great pleasure for me to acknowledge my indebtedness to Professor Nils Svartholm for his constant encouragement and sage advise. I would also like to thank Dr Arne Kihlberg for a pleasant co-operation and for many stimulating discussions. It is also appropriate to thank here all other members of the staff of the Institute of Theoretical Physics in Göteborg, who all, in different ways, have helped to create the inspiring research atmosphere in which I have had the privilege to work.

Institute of Theoretical Physics
Göteborg, April 1967

Staffan Ström

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GÖTEBORG
ELANDERS BOKTRYCKERI AKTIEBOLAG
1967

ARKIV FÖR FYSIK

UTGIVET AV

KUNGL. SVENSKA VETENSKAPSAKADEMIEN

Band 29 nr 39

STAFFAN STRÖM

On the matrix elements of a unitary representation of the homogeneous Lorentz group

ALMQVIST & WIKSELL

STOCKHOLM

GÖTEBORG · UPPSALA

1965

ARKIV FÖR FYSIK Band 29 nr 39

3.650511 Communicated 10 March 1965 by I. WALLER and E. RUDBERG

On the matrix elements of a unitary representation of the homogeneous Lorentz group

By STAFFAN STRÖM

ABSTRACT

The physically interesting unitary irreducible representations of the homogeneous Lorentz group are characterized by two numbers, l_0 and ν. In the present work we derive the form of the matrix elements of a finite Lorentz transformation in a unitary representation (l_0, ν) by integration of the infinitesimal relations. It is shown that the matrix elements in a number of cases satisfy a second order differential equation, whose solution can be expressed in terms of hypergeometric polynomials. In particular it is shown, that it is always possible to find a column, whose elements can be written in this way, and thereby we have obtained simple and explicit expressions for a set of functions, which forms a basis for a linear representation of the homogeneous Lorentz group. The general matrix element, which satisfies a fourth order differential equation, is determined by an application of the ladder operator technique.

1. Introduction

The unitary representations of the homogeneous Lorentz group (HLG) are infinite-dimensional. They are characterized by two numbers, l_0 and ν, where l_0 is a non-negative integer or half-integer while ν is an arbitrary real number [1]. If $l_0=0$, then ν can also be a purely imaginary number, such that $\nu=i\lambda$ with $0<\lambda<1$, but these representations do not seem to be used in physics and they will not be discussed here. A unitary representation characterized by (l_0, ν) contains the unitary representations of the threedimensional rotation group corresponding to the weights $l_0, l_0+1, \dots$ and contains each of these exactly once.

Previously the matrix elements of a finite Lorentz transformation in a unitary representation (l_0, ν) have been studied by Dolginov *et al.* [2, 3, 4]. These authors start from the known matrix elements of a rotation in a unitary representation of the fourdimensional rotation group. The corresponding functions in case of Lorentz metric are then obtained by analytic continuation. In the present work we use a more direct method, that is we derive the form of the matrix elements by integration of the infinitesimal relations. In the rest of this section we will give a number of well-known properties of the HLG and its unitary representations, which will be used later in the paper.

The coordinates of the Lorentz space are denoted by x_i, $i=0, 1, 2, 3$ where x_0 is the time coordinate. The infinitesimal generators are [1]

$$J_{23}=M_1,\ J_{31}=M_2,\ J_{12}=M_3,$$
$$J_{01}=N_1,\ J_{02}=N_2,\ J_{03}=N_3, \tag{1.1}$$

and we use the usual vector notation $\mathbf{M}=(M_1, M_2, M_3)$, $\mathbf{N}=(N_1, N_2, N_3)$. The commutation relations are

$$[M_r, M_s]=iM_t,\ [M_r, N_r]=0,\ [N_r, N_s]=-iM_t,$$
$$[M_r, N_s]=[N_r, M_s]=iN_t,\ (r, s, t \text{ in cyclic perm.}). \tag{1.2}$$

Every proper homogeneous Lorentz transformation L may be written [1]

$$L=R_1\cdot A\cdot R_2, \tag{1.3}$$

where R_1 and R_2 are threedimensional rotations and A is an acceleration in a certain prescribed direction. This direction can be chosen arbitrarily. The HLG can be described by six independent parameters. We choose Euler angles as parameters for the threedimensional rotations and according to (1.3) we can then write

$$L=R_z(\varphi)\,R_x(\theta)\,R_z(\psi')\,A_z(\alpha)\,R_z(\psi'')\,R_x(\beta)\,R_z(\gamma), \tag{1.4}$$

where $A_z(\alpha)$ is an acceleration in the z-direction with a velocity $v=\mathrm{Tgh}\,\alpha(c=1)$. Since

$$A_z(\alpha)\,R_z(\psi'')=R_z(\psi'')\,A_z(\alpha)$$

and

$$R_z(\psi')\,R_z(\psi'')=R_z(\psi'+\psi'')$$

only $\psi'+\psi''=\psi$ enters as a parameter into L and we have:

$$L=L(\varphi, \theta, \psi, \alpha, \beta, \gamma)=R_z(\varphi)\,R_x(\theta)\,R_z(\psi)\,A_z(\alpha)\,R_x(\beta)\,R_z(\gamma). \tag{1.5}$$

Denote the matrices of a unitary representation of the HLG by T:

$$T(L)=T(\varphi, \theta, \psi, \alpha, \beta, \gamma)$$
$$T(L_1)\,T(L_2)=T(L_1\cdot L_2).$$

If we now choose L_1 as an infinitesimal rotation in the different coordinate planes and then expand the right-hand side in a Taylor series around L_2 we get the differential operators which, acting upon the matrix elements, represent the generators M_i and N_i. These are given in Appendix 1.

We will now consider the matrix elements $T^{ll'}_{mm'}$ of the operator $O(L)$, representing L, in a basis in which $\mathbf{M}^2$ and M_3 are diagonal. According to the decomposition (1.5) the matrix elements may be written as follows

$$T^{ll'}_{mm'}(\varphi, \theta, \psi, \alpha, \beta, \gamma;\, l_0, \nu)=\sum_{n=-\min(l,l')}^{n=\min(l,l')} D^l_{m,n}(\varphi, \theta, \psi)\,A^{ll'}_n(\alpha;\, l_0; \nu)\,D^{l'}_{n,m'}(0, \beta, \gamma), \tag{1.6}$$

where l, m are row indices and l', m' column indices. The functions $D^l_{m,n}$ are the known matrix elements of a $(2+1)$-dimensional representation of the three-dimensional rotation group and the functions $A^{ll'}_n(\alpha; l_0, \nu)$ are new functions to be determined. An arbitrary column of the matrix $T^{ll'}_{mm'}$ acts as a set of basis functions and therefore they shall satisfy the following relations [1] (and we hereby make a certain choice of phases):

$$M_+T^{l,l'}_{m,m'} = \alpha^l_{m,m+1}T^{l,l'}_{m+1,m'},$$

$$M_-T^{l,l'}_{m,m'} = \alpha^l_{m-1,m}T^{l,l'}_{m-1,m'},$$

$$M_3T^{l,l'}_{m,m'} = mT^{l,l'}_{m,m'},$$

$$N_+T^{l,l'}_{m,m'} = i\alpha^{l+1}_{-m,m+1}\gamma(l+1; l_0, \nu)T^{l+1,l'}_{m+1,m'} + \frac{\nu l_0}{l(l+1)}\alpha^l_{m,m+1}T^{l,l'}_{m+1,m} + i\alpha^l_{m+1,-m}\gamma(l; l_0, \nu)T^{l-1,l'}_{m+1,m'},$$

$$N_-T^{l,l'}_{m,m'} = -i\alpha^{l+1}_{m-1,-m}\gamma(l+1; l_0, \nu)T^{l+1,l'}_{m-1,m'} + \frac{\nu l_0}{l(l+1)}\alpha^l_{m-1,m}T^{l,l'}_{m-1,m'} - i\alpha^l_{-m,m-1}\gamma(l; l_0, \nu)T^{l-1,l'}_{m-1,m'},$$

$$N_3T^{l,l'}_{m,m'} = -i\alpha^{l+1}_{m,m}\gamma(l+1; l_0, \nu)T^{l+1,l'}_{m,m'} + \frac{ml_0\nu}{l(l+1)}T^{l,l'}_{m,m'} + i\alpha^l_{m,m}\gamma(l; l_0, \nu)T^{l-1,l'}_{m,m'}, \tag{1.7}$$

where $\quad M_\pm = M_1 \pm iM_2, \;\; N_\pm = N_1 \pm iN_2,$

$$\gamma(l; l_0, \nu) = \frac{1}{l}\sqrt{\frac{(l^2-l_0^2)(l^2+\nu^2)}{(2l+1)(2l-1)}}, \qquad \alpha^l_{m,n} = \sqrt{(l-m)(l+n)}.$$

The HLG has two independent invariants. These can be chosen as $\mathbf{M}^2-\mathbf{N}^2$ and $\mathbf{M}\cdot\mathbf{N}$. In an irreducible representation $\mathbf{M}^2-\mathbf{N}^2$ and $\mathbf{M}\cdot\mathbf{N}$ are therefore scalar operators and in a representation (l_0, ν) their values are $l_0^2-1-\nu^2$ and $l_0\cdot\nu$ respectively. This means that the matrix elements $T^{ll'}_{mm'}$ satisfy the following differential equations:

$$(\mathbf{M}^2-\mathbf{N}^2)T^{l,l'}_{m,m'} = (l_0^2-1-\nu^2)T^{l,l'}_{m,m'}, \qquad \mathbf{M}\cdot\mathbf{N}\,T^{l,l'}_{m,m'} = l_0\cdot\nu T^{l,l'}_{m,m'}. \tag{1.8}$$

The differential operators corresponding to $\mathbf{M}^2-\mathbf{N}^2$ and $\mathbf{M}\cdot\mathbf{N}$ are given in Appendix 1.

2. Differential aquations for $A^{ll'}_n(\alpha; l_0, \nu)$

In this section we will list a number of relations for the functions $A^{ll'}_n(\alpha; l_0, \nu)$ which follow from (1.7) and (1.8). In the expression (1.6) for $T^{ll'}_{mm'}$ the D-functions are the well-known matrix elements of a unitary representation of the threedimensional rotation group. These functions satisfy a number of recursion relations, difference-differential equations etc. (See e.g. ref. [5]). These relations together with the orthogonality relations for the D-functions can now be used to extract from equations (1.7) and (1.8) relations which involve only the functions $A_n^{ll'}(\alpha; l_0, \nu)$. First we use (1.7) and get three relations which contain first order derivatives with respect to α and differences in both indices l and n. From (1.8) we then get only one new relation, which contains a second order derivative with respect to α and differences only in the index n. The second of the equations (1.8) gives a relation containing first order derivatives with respect to α and is identical to one of the relations which we get from (1.7). We may remark, however, that the equations (1.8) are sufficient for a determination of the *form* of the functions $A_n^{ll'}(\alpha; l_0, \nu)$, but to get the correct normalization we have to use (1.7). We now observe the important fact that from the relations obtained by the procedure indicated above we can form ladder operators acting on the indices l and n separately, i.e. operators, which increase or decrease one of these indices by one unit.

We now give the explicit form of the above-mentioned relations. From (1.7) we have:

$$\sqrt{(l+n+1)(l-n+1)}\left[\frac{d}{d\alpha} - l \operatorname{Coth}\alpha\right] A_n^{l,l'}(\alpha; l_0, \nu)$$

$$= -\sqrt{((l+1)^2 - l_0^2)((l+1)^2+\nu^2)}\sqrt{\frac{2l+1}{2l+3}}A_n^{l+1,l'}(\alpha; l_0\,\nu)$$

$$-\frac{1}{2\operatorname{Sh}\alpha}[\sqrt{(l-n)(l-n+1)(l'-n)(l'+n+1)}\,A_{n+1}^{l,l'}(\alpha; l_0, \nu)$$

$$+\sqrt{(l+n)(l+n+1)(l'+n)(l'-n+1)}\,A_{n-1}^{l,l'}(\alpha; l_0, \nu)]. \tag{2.1}$$

$$\sqrt{(l+n)(l-n)}\left[\frac{d}{d\alpha} + (l+1)\operatorname{Coth}\alpha\right] A_n^{l,l'}(\alpha; l_0, \nu)$$

$$= \sqrt{(l^2 - l_0^2)(l^2+\nu^2)}\sqrt{\frac{2l+1}{2l-1}}A_n^{l-1,l'}(\alpha; l_0, \nu)$$

$$+\frac{1}{2\operatorname{Sh}\alpha}[\sqrt{(l+n)(l+n+1)(l'+n+1)(l'-n)}\,A_{n+1}^{l,l'}(\alpha; l_0, \nu)$$

$$+\sqrt{(l-n)(l-n+1)(l'+n)(l'-n+1)}\,A_{n-1}^{l,l'}(\alpha; l_0, \nu)]. \tag{2.2}$$

$$n\left[\frac{d}{d\alpha} + \operatorname{Coth}\alpha\right] A_n^{l,l'}(\alpha; l_0, \nu) = -il_0\,\nu A_n^{l,l'}(\alpha; l_0, \nu)$$

$$+\frac{1}{2\operatorname{Sh}\alpha}[\sqrt{(l+n)(l-n+1)(l'+n)(l'-n+1)}\,A_{n-1}^{l,l'}(\alpha; l_0, \nu)$$

$$-\sqrt{(l+n+1)(l-n)(l'+n+1)(l'-n)}\,A_{n+1}^{l,l'}(\alpha; l_0, \nu)]. \tag{2.3}$$

The ladder operators are found to be:

$$L_+(n)A_n^{l,l'}(\alpha;l_0,\nu)$$

$$=\left[\frac{\operatorname{Sh}^2\alpha}{\operatorname{Ch}\alpha}\frac{d^2}{d\alpha^2}+2(n+1)\operatorname{Sh}\alpha\frac{d}{d\alpha}+2n\operatorname{Ch}\alpha+2il_0\nu\operatorname{Sh}\alpha+\frac{\operatorname{Sh}^2\alpha}{\operatorname{Ch}\alpha}\{n^2-l_0^2+1+\nu^2\}\right.$$

$$\left.+\frac{1}{\operatorname{Ch}\alpha}\{2n^2-l(l+1)-l'(l'+1)\}\right]A_n^{l,l'}(\alpha;l_0,\nu)$$

$$=-2\sqrt{(l-n)(l+n+1)(l'-n)(l'+n+1)}\,A_{n+1}^{l,l'}(\alpha;l_0,\nu). \tag{2.4}$$

$$L_-(n)A_n^{l,l'}(\alpha;l_0,\nu)$$

$$=\left[\frac{\operatorname{Sh}^2\alpha}{\operatorname{Ch}\alpha}\frac{d^2}{d\alpha^2}-2(n-1)\operatorname{Sh}\alpha\frac{d}{d\alpha}-2n\operatorname{Ch}\alpha-2il_0\nu\operatorname{Sh}\alpha+\frac{\operatorname{Sh}^2\alpha}{\operatorname{Ch}\alpha}\{n^2-l_0^2+1+\nu^2\}\right.$$

$$\left.+\frac{1}{\operatorname{Ch}\alpha}\{2n^2-l(l+1)-l'(l'+1)\}\right]A_n^{l,l'}(\alpha;l_0,\nu)$$

$$=-2\sqrt{(l+n)(l-n+1)(l'+n)(l'-n+1)}\,A_{n-1}^{l,l'}(\alpha;l_0,\nu). \tag{2.5}$$

$$K_+(l)A_n^{l,l'}(\alpha;l_0,\nu)$$

$$=\left[\frac{\operatorname{Sh}\alpha}{\operatorname{Ch}\alpha}\frac{d^2}{d\alpha^2}-2l\frac{d}{d\alpha}+2(l(l+1)-n^2)\frac{\operatorname{Ch}\alpha}{\operatorname{Sh}\alpha}-\frac{2nil_0\nu}{(l+1)}+\frac{\operatorname{Sh}\alpha}{\operatorname{Ch}\alpha}\{n^2-l_0^2+1+\nu^2\}\right.$$

$$\left.+\frac{1}{\operatorname{Sh}\alpha\operatorname{Ch}\alpha}\{2n^2-l(l+1)-l'(l'+1)\}\right]A_n^{l,l'}(\alpha;l_0,\nu)$$

$$=-\frac{2}{(l+1)}\sqrt{((l+1)^2-n^2)((l+1)^2-l_0^2)((l+1)^2+\nu^2)}\sqrt{\frac{2l+1}{2l+3}}\,A_n^{l+1,l'}(\alpha;l_0,\nu). \tag{2.6}$$

$$K_-(l)A_n^{l,l'}(\alpha;l_0,\nu)$$

$$=\left[\frac{\operatorname{Sh}\alpha}{\operatorname{Ch}\alpha}\frac{d^2}{d\alpha^2}+2(l+1)\frac{d}{d\alpha}+2(l(l+1)-n^2)\frac{\operatorname{Ch}\alpha}{\operatorname{Sh}\alpha}+\frac{2nil_0\nu}{l}+\frac{\operatorname{Sh}\alpha}{\operatorname{Ch}\alpha}\{n^2-l_0^2+1+\nu^2\}\right.$$

$$\left.+\frac{1}{\operatorname{Sh}\alpha\operatorname{Ch}\alpha}\{2n^2-l(l+1)-l'(l'+1)\}\right]A_n^{l,l'}(\alpha;l_0,\nu)$$

$$=\frac{2}{l}\sqrt{(l^2-l_0^2)(l^2-n^2)(l^2+\nu^2)}\sqrt{\frac{2l+1}{2l-1}}\,A_n^{l-1,l'}(\alpha;l_0,\nu). \tag{2.7}$$

We can now form a pure differential equation for $A_n^{ll'}(\alpha;l_0,\nu)$, which we will refer to later. From (2.4) and (2.5) we find the following fourth order differential equation for $A_n^{ll'}(\alpha;l_0,\nu)$ (exactly the same equation can, of course, also be derived from (2.6) and (2.7)):

$$L_+(n+1)L_-(n)A_n^{l,l'}(\alpha;l_0,\nu)=4\beta^2(n+1)A_n^{l,l'}(\alpha;\mathrm{l}_0,\nu),$$

where $$\beta(n)=\sqrt{(l+n)(l-n+1)(l'+n)(l'-n+1)}. \tag{2.8}$$

In the variable $v = \operatorname{Tgh}\alpha$ this equation has rational coefficents. It is given in detail in Appendix 2. We observe that it is left unchanged by the following substitutions:

$$\begin{aligned}(l, l', n, v, l_0, \nu) &\rightarrow (l', l, n, v, l_0, \nu),\\ &\rightarrow (l, l', l_0, v, n, \nu),\\ &\rightarrow (l, l', n, v, -l_0, -\nu),\\ &\rightarrow (l, l', -n, v, l_0, -\nu),\\ &\rightarrow (l, l', n, -v, l_0, -\nu),\\ &\rightarrow (-l-1, l', n, v, l_0, \nu).\end{aligned} \tag{2.9}$$

Singular points are -1, 0, 1 and ∞. The exponents λ, γ and δ in

$$A_n^{l,l'}(\alpha; l_0, \nu) = v^{\lambda}(1-v)^{\gamma}(1+v)^{\delta}\, Q_n^{ll'}(v; l_0, \nu, \lambda, \gamma, \delta)$$

are found to be

$$\begin{array}{lll}\lambda_1 = l' - l, & \gamma_1 = \tfrac{1}{2}(n+1+l_0-i\nu), & \delta_1 = \tfrac{1}{2}(-n+1+l_0-i\nu),\\ \lambda_2 = l - l', & \gamma_2 = \tfrac{1}{2}(n+1-l_0+i\nu), & \delta_2 = \tfrac{1}{2}(-n+1-l_0+i\nu),\\ \lambda_3 = l' + l + 1, & \gamma_3 = \tfrac{1}{2}(-n+1+l_0+i\nu), & \delta_3 = \tfrac{1}{2}(n+1+l_0+i\nu),\\ \lambda_4 = -l' - l - 1, & \gamma_4 = \tfrac{1}{2}(-n+1-l_0-i\nu), & \delta_4 = \tfrac{1}{2}(n+1-l_0-i\nu).\end{array} \tag{2.10}$$

If λ_1 and γ_1 are given the remaining λ's, γ's and δ's are determined by the properties (2.9). For $v=0$ the Lorentz transformation is a pure rotation and therefore $A_n^{ll'}(\alpha; l_0, \nu)$ must have the property that

$$A_n^{l,l'}(0; l_0, \nu) = \delta_{ll'},$$

since the representation then decomposes into a direct sum of representations of the three-dimensional rotation group. This means that for $l \leqslant l'$ we have to choose a solution corresponding to the exponent λ_1 whereas for $l' < l$ we have to choose a solution corresponding to λ_2.

We now put formally

$$Q_n^{ll'}(v; l_0, \nu, \lambda, \gamma, \delta) = \sum_{k=0}^{\infty} p_k(n)\, v^k.$$

From (2.5) we get a linear relation between $p_k(n)$, $p_{k-1}(n)$, $p_{k-2}(n)$ and $p_k(n+1)$. In the same way (2.6) gives a linear relation between $p_k(n) \dots p_{k-4}(n)$ and $p_k(n-1)$. We thus get a linear relation between seven consecutive coefficients $p_k(n)$ for fixed n. Obviously this relation also follows from the fourth order equation for $Q_n^{ll'}$.

3. Determination of $A_l^{ll'}(\alpha; l_0, \nu)$

To determine $A_l^{ll'}(\alpha; l_0, \nu)$ we will make use of the fact that for $n=l$ the r.h.s. of equation (2.4) is equal to zero, i.e. $A_l^{ll'}(\alpha, l_0, \nu)$ with $l \leqslant l'$ satisfies the following second order differential equation:

$$L_+(l)\,A_l^{l,l'}(\alpha;l_0,\nu)=0, \tag{3.1}$$

which turns out to be essentially a hypergeometric equation. Similar equations are satisfied by $A_{-l}^{ll'}(\alpha;l_0,\nu)\,(l\leqslant l')$ and by $A_n^{l_0l'}(\alpha;l_0,\nu)\,(|n|\leqslant l_0)$ (cf. (2.5) and (2.6)). This will be used in the next section.

After a change to the variable $y=e^{-2\alpha}$, equation (3.1) reads explicitly

$$\left[y(1-y)\frac{d^2}{dy^2}+(1-y-(l+1)(1+y))\frac{d}{dy}+\frac{l}{2}\frac{(1+y)^2}{y(1-y)}+il_0\nu\frac{1+y}{y}\right.$$

$$\left.+(l^2-l_0^2+1+\nu^2)\frac{1-y}{4y}+\frac{1}{1-y}(l(l-1)-l'(l'+1))\right]A_l^{l,l'}(\alpha;l_0,\nu)=0. \tag{3.2}$$

Put
$$A_l^{l,l'}(\alpha;l_0,\nu)=y^{\sigma}(1-y)^{\varrho}B_l^{ll'}(y;l_0,\nu)$$
and
$$\varrho_1=\tfrac{1}{2}(l+1+l_0-i\nu),\quad \sigma_1=l'-l,$$
$$\varrho_2=\tfrac{1}{2}(l+1-l_0+i\nu),\quad \sigma_2=-l'-l-1.$$

If we choose (ϱ,σ) as (ϱ_1,σ_1), (ϱ_2,σ_1), (ϱ_1,σ_2) or (ϱ_2,σ_2) the equation for $B_l^{ll'}(y,l_0,\nu)$ becomes

$$\left[y(1-y)\frac{d^2}{dy^2}+(\gamma-(1+\alpha+\beta)\,y)\frac{d}{dy}-\alpha\cdot\beta\right]B_l^{ll'}(y;l_0,\nu)=0,$$

where
$$\alpha=\varrho+\sigma+\tfrac{1}{2}(l+1+l_0+i\nu),$$
$$\beta=\varrho+\sigma+\tfrac{1}{2}(l+1-l_0-i\nu),$$
$$\gamma=2\varrho-1.$$

This is an hypergeometric equation. Since $y=1$ corresponds to $v=0$, we seek a solution which is bounded for $y=1$. We now distinguish between two cases:

I. $\nu\neq 0$, l_0 *integer or half-integer or* $\nu=0$, l_0 *half-integer*

In this case we can obtain a solution corresponding to $\sigma=\sigma_1$, by forming a linear combination of two solutions corresponding to $\sigma=\sigma_2$. When $\nu\neq 0$, l_0 integer or half-integer or $\nu=0$, l_0 half-integer, two linearly independent solutions of (3.2) are $C_l^{ll'}(y,l_0,\nu)$ and $C_l^{ll'}(y,-l_0,-\nu)$ where

$$C_l^{ll'}(y;l_0,\nu)=y^{\frac{1}{2}(l+1+l_0-i\nu)}(1-y)^{-l'-l-1}F(-l'+l_0,-l'-i\nu,1+l_0-i\nu,y). \tag{3.3}$$

The hypergeometric functions which occur in $C_l^{ll'}(y,l_0,\nu)$ and $C_l^{ll'}(y,-l_0-\nu)$ are polynomials of degree $l'-l_0$ and $l'+l_0$ respectively. As we want to consider the behaviour in the neighbourhood of $y=1$, we change to the variable $1-y$. If $F(\alpha,\beta,\gamma,y)$ is a polynomial in y, this polynomial can also be written as const. $F(\alpha,\beta,-\gamma+1+\alpha+\beta,1-y)$. In this way we find that a solution to (3.2) which for $y=1$ has the exponent $l'-l$ can be written:

$$D_l^{ll'}(y;l_0,\nu)=y^{\frac{1}{2}(l+1)}(1-y)^{-l'-l-1}[y^{\frac{1}{2}(l_0-i\nu)}F(-l'+l_0,\,-l'-i\nu,\,-2l',\,1-y)$$
$$-y^{-\frac{1}{2}(l_0-i\nu)}F(-l'+i\nu,\,-l'-l_0,\,-2l',\,1-y)]. \tag{3.4}$$

Apart from a normalization constant the function $A_l^{ll'}(\alpha;l_0,\nu)$ is equal to $D_l^{ll'}(y,l_0,\nu)$. We put

$$A_l^{l,l'}(\alpha:l_0,\nu)=N(l,l';l_0,\nu)\,D_l^{ll'}(y;l_0,\nu).$$

Using e.g. Saalschutz theorem we find that

$$D_{l'}^{l'l'}(1,l_0,\nu)=(-1)^{l'+l_0+1}\frac{(l'-l_0)!\,(l'+l_0)!\,(-l'-i\nu)_{2l'+1}}{(2l'+1)!\,(2l')!}, \tag{3.5}$$

where $(p)_r=p(p+1)\dots(p+r-1)$. If we put $n=l+1$ in (2.1) we get a relation between $A_l^{ll'}$ and $A_{l+1}^{l+1,l'}$ and repeated use of that relation gives us

$$N(l,l';l_0,\nu)=\frac{(-1)^{2l'+1+l_0-l}}{(-l'-i\nu)_{2l'+1}}\sqrt{\frac{(2l'+1)!\,(2l+1)!\,(l'+l)!\,(2l')!\,((l+1)^2+\nu^2)\dots(l'^2+\nu^2)}{(l'+l_0)!\,(l'-l_0)!\,(l+l_0)!\,(l-l_0)!\,(l'-l)!}}.$$

We observe that for $l'=l_0$ we get in (3.4) a hypergeometric function $F(-l'+i\nu,\,-2l',\,-2l',\,1-y)$ which then stands for the polynomial of degree $2l'$ which we get from the hypergeometric series in the usual way. (This polynomial is the beginning of the expansion of $(1-(1-y))^{l'-i\nu}$.)

For $l_0=0$ we may write the solution of (3.5) in the following way (cf. 2)):

$$\text{const}\,(\text{Sh}\,\alpha)^{l'-l}\frac{d^{l'+1}(\cos\nu\alpha)}{d(\text{Ch}\,\alpha)^{l'+1}}. \tag{3.6}$$

(To see this, show that (3.6) and $A_l^{ll'}(\alpha,0,\nu)$ satisfy the same recursion relations.) There is no strict generalization of (3.6) to the case $l_0\neq 0$. It is possible, however, to derive an expression for $A_l^{ll'}(\alpha,l_0,\nu)$ in which derivatives occur in a way which is somewhat similar to the way they occur in (3.6). To achieve this we use the following connection between the Jacobi polynomials and the hypergeometric polynomials:

$$P_n^{(\alpha,\beta)}(x)=\binom{2n+\alpha+\beta}{n}\left(\frac{x-1}{2}\right)^n F\left(-n,\,-n-\alpha,\,-2n-\alpha-\beta,\,\frac{2}{1-x}\right).$$

We further have the relations

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}]$$

and
$$P_n^{(\alpha,\beta)}(x)=(-1)^n\cdot P_n^{(\beta,\alpha)}(-x).$$

From $1-y=2/(1+x)$ we have $x=(1+y)/(1-y)=\text{Coth}\,\alpha$. In this way we get a representation of $A_l^{ll'}(\alpha,l_0,\nu)$ where we have derivatives with respect to $\text{Coth}\,\alpha$

whereas in the special case of $l_0=0$ we could use derivatives with respect to $\operatorname{Ch}\alpha$. In order to get the argument $\operatorname{Ch}\alpha$ in the Jacobi polynomials we must first make a quadratic transformation of the argument of the hypergeometric functions. This gives new hypergeometric functions only if $\pm(1-\gamma)$, $\pm(\alpha-\beta)$ $\pm(\alpha+\beta-\gamma)$ have the property that one of them is equal to $\frac{1}{2}$ or that two are equal. This is not the case when $l_0\neq 0$, $\nu\neq 0$. The complication which arises if the variable $\operatorname{Ch}\alpha$ is used for the case $l_0\neq 0$, $\nu\neq 0$ is also seen from the fact that in this variable (3.2) no longer has rational coefficients.

In terms of Jacobi polynomials the final result for the case $\nu\neq 0$, l_0 integer or $\nu=0$, l_0 half-integer and $l\leqslant l'$ is that

$$\begin{aligned}A_l^{ll'}(\alpha; l_0, \nu) &= N(l, l'; l_0, \nu)\cdot 2^{-l-1}\binom{2l'}{l'-l_0}^{-1}[2^{-l_0}(x-1)^{\frac{1}{2}(l+1+l_0-i\nu)}(x+1)^{\frac{1}{2}(l+1+l_0+i\nu)}P_{l'-l_0}^{l_0-i\nu,\, l_0+i\nu}(x)\\ &\quad -2^{l_0}(x-1)^{\frac{1}{2}(l+1-l_0+i\nu)}(x+1)^{\frac{1}{2}(l+1-l_0-i\nu)}P_{l'+l_0}^{-l_0+i\nu,\, -l_0-i\nu}(x)]. \qquad (3.7)\end{aligned}$$

II. $\nu=0$, l_0 *integer*

For $\nu=0$, l_0 integer the r.h.s. of (3.4) is identically zero. The solution may now be expressed as

$$A_l^{ll'}(\alpha; l_0, 0) = Ny^{\frac{1}{2}(l+l_0+1)}(1-y)^{l'-l}F(l'+1+l_0, l'+1, 2l'+2, 1-y). \qquad (3.8)$$

Here we have $\gamma=2\beta$ in the hypergeometric function and it can therefore be expressed in Legendre functions and for these we may choose the argument $\operatorname{Ch}\alpha$. We get

$$A_l^{ll'}(\alpha; l_0, 0) = N'(\operatorname{Sh}\alpha)^{-(l+\frac{1}{2})}P_{-l_0-\frac{1}{2}}^{-l'-\frac{1}{2}}(\operatorname{Ch}\alpha). \qquad (3.9)$$

We note that analogous expressions may be used for the case $l_0=0$, $\nu\neq 0$. (Make the substitution $l_0\to -i\nu$ in (3.8) and (3.9).)

4. The set of functions $T_{mm'}^{ll_0}$, $l\geqslant l_0$, $|m|\leqslant l_0$, l_0, m' fixed

As noted above the elements of an arbitrary column in the matrix $T_{mm'}^{ll'}$ (i.e. the functions $T_{mm'}^{ll'}$ with l', m' fixed) form a basis for a linear representation of the HLG. However, they do not form a Hilbert space, since they are not square-integrable (cf. [4]; for an explicit realization of a representation space as a Hilbert space and a basis in this space see ref. [1]). It is therefore of interest to consider in some detail this set of functions. Different choices of l' and m' give different sets of functions, which, however, transform in exactly the same way and it is clear that we get the most simple set of functions by putting $l'=l_0$ and $m'=0$ or $m'=\frac{1}{2}$ according to whether l_0 is integer or half-integer. According to (1.6) $T_{mm'}^{ll_0}$ is a sum of functions $A_n^{ll_0}(\alpha; l_0, \nu)$ and these functions are therefore of a special interest. We shall see that they can be given in closed form. Note that the ladder operator K_- gives zero according to the following rules:

$$K_-(n)\,A_n^{n,l'}(\alpha; l_0, \nu) = 0 \quad \text{if} \quad n \geqslant l_0, \tag{4.1}$$

$$K_-(l_0)\,A_n^{l_0,l'}(\alpha; l_0, \nu) = 0 \quad \text{if} \quad n \leqslant l_0, \tag{4.2}$$

(4.2) turns out to be identical with

$$L_+(l_0)\,A_{l_0}^{l_0,l'}(\alpha; n, \nu) = 0. \tag{4.3}$$

The additional restriction that $n \geqslant 0$ must then be imposed, however. We note that we have the following symmetry in l and l':

$$A_n^{l,l'}(\alpha; l_0, \nu) = (-1)^{l'-l} A_n^{l',l}(\alpha; l_0, \nu), \tag{4.4}$$

(cf. equation (5.5) below). The following symmetry in n and ν is also useful

$$A_n^{l,l'}(\alpha; l_0, \nu) = A_{-n}^{l,l'}(\alpha; l_0, -\nu). \tag{4.5}$$

Taking (4.3) and (4.4) into account we thus get

$$\begin{aligned} A_n^{ll_0}(\alpha; l_0\, \nu) = N'' y^{\frac{1}{2}(l_0+1)}(1-y)^{-l_0-l-1}\,[&y^{\frac{1}{2}(n-i\nu)}F(-l+n, -l-i\nu, -2l, 1-y) \\ &- y^{-\frac{1}{2}(n-i\nu)}F(-l+i\nu, -l-n, -2l, 1-y)]. \end{aligned} \tag{4.6}$$

Because of the property (4.5) it is now easy to remove the restriction on n.

When $l_0 = 0$ we can choose

$$T_{m,0}^{l,0} = D_{m,0}^{l}(\varphi, \theta; 0)\,A_0^{l,0}(\alpha; 0, \nu)$$

as basis functions. These may be regarded as functions of a fourvector of unit length $x = (x_0, \mathrm{x}) = (\mathrm{Ch}\,\alpha,\ \mathrm{Sh}\,\alpha \sin\theta\cos\varphi,\ \mathrm{Sh}\,\alpha\sin\theta\sin\varphi,\ \mathrm{Sh}\,\alpha\cos\theta)$.

When $l_0 > 0$ is an integer the simplest functions are

$$T_{m,0}^{l,l_0} = \sum_{n=-l_0}^{n=l_0} D_{m,n}^{l}(\varphi, \theta, \psi)\,A_n^{l,l_0}(\alpha; l_0, \nu)\,D_{n,0}^{l_0}(0, \beta, 0).$$

Compared to the case $l_0 = 0$ they are functions of two more variables. Finally we get for the case of l_0 half-integer the functions

$$T_{m,\frac{1}{2}}^{l,l_0} = \sum_{n=-l_0}^{n=l_0} D_{m,n}^{l}(\varphi, \theta, \psi)\,A_n^{l,l_0}(\alpha; l_0, \nu)\,D_{n,\frac{1}{2}}^{l_0}(0, \beta, \gamma),$$

which depend on all the parameters of the HLG.

5. Determination of $A_n^{ll'}(\alpha; l_0, \nu)$

We now turn to the determination of the general function $A_n^{ll'}(\alpha; l_0, \nu)$. We have not succeeded in giving a "closed" form in this case but in the present section we will describe an iteration procedure by which $A_n^{ll'}(\alpha; l_0, \nu)$ can be determined. In particular the structure of $A_n^{ll'}(\alpha; l_0, \nu)$ in the variable $v = \mathrm{Tgh}\,\alpha$ will be given.

I. $\nu \neq 0$, l_0 *integer or half-integer or* $\nu = 0$, l_0 *half-integer*

In section 3 we determined the functions $A_l^{ll'}$ for $l \leqslant l'$. From $A_l^{ll'}(\alpha; l_0, \nu)$ we can get $A_n^{ll'}(\alpha; l_0, \nu)$ by means of a repeated application of either (2.3) or (2.5). We prefer to use (2.3). This equation contains the operator $2n[\operatorname{Sh}\alpha(d/d\alpha) + \operatorname{Ch}\alpha] + 2il_0\nu \operatorname{Sh}\alpha$, which we denote by $O(n)$. Repeated use of (2.3) gives an expression for $A_n^{ll'}(\alpha; l_0, \nu)$ which contains $O(n)\, A_l^{ll'}(\alpha; l_0, \nu)$. Using the properties of the Jacobi polynomials we find:

$$O(n)\, D_l^{ll'}\left(\frac{x+1}{x-1}; l_0, \nu\right) = -2n(l'+l)\, D_{l-1}^{l-1,l'+1}\left(\frac{x+1}{x-1}; l_0, \nu\right) + il_0\nu\, \frac{l'(l'+1) - n(l-1)}{l'(l'+1)}\, D_{l-1}^{l-1,l'}\left(\frac{x+1}{x-1}; l_0, \nu\right) + \frac{n(l'^2+\nu^2)(l'+1-l)(l'^2-l_0^2)}{2(l')^2(2l'+1)(2l'-1)}\, D_{l-1}^{l-1,l'-1}\left(\frac{x+1}{x-1}; l_0, \nu\right). \quad (5.1)$$

A product of several operators $O(n)$ acting upon $D_l^{ll'}$ may then give functions $D_{l-k}^{l-k,l'+\mu}$ where $l-k < l_0$. To these D-functions there is no corresponding function $A = N \cdot D$ (cf. section 3). Further we note that we always have $l' + \mu \geqslant l_0$. This follows from the fact that the coefficient for $D_{l-1}^{l-1,l'}(x+1/(x-1); l_0, \nu)$ in (5.1) contains a factor $(l' - l_0)$. Consequently this term disappears for $l' = l_0$.

In the following calculation it is convenient to use the symbol $\beta(n)$, which we introduced in equation (2.8). Equation (2.3) then reads

$$A_{n-1}^{l,l'} = \frac{O(n)}{\beta(n)}\, A_n^{l,l'} + \frac{\beta(n+1)}{\beta(n)}\, A_{n+1}^{l,l'}.$$

We further simplify the notation by putting

$$\frac{O(n)}{\beta(n)} = L_k \text{ and } \frac{\beta(n+1)}{\beta(n)} = \gamma_k, \text{ where } n = l-k \text{ (thus } \gamma_0 = 0).$$

We thus have
$$A_{l-(k+1)}^{l,l'} = L_k\, A_{l-k}^{l,l'} + \gamma_k\, A_{l-(k-1)}^{l,l'}. \quad (5.2)$$

It is now easy to find the general structure of the expression for $A_{l-k}^{ll'}$ which is obtained by repeated application of (5.2). The operator which, acting upon $A_l^{ll'}$, gives $A_{l-k}^{ll'}$ will contain $\binom{k-p}{p}$ terms with a product of $(k-2p)$ operators L_i $(p = 0, 1, 2, \ldots, k \geqslant 2p)$. A term containing $(k-2p)$ L-operators contains p factors γ. The indices of L_i and γ_j are determined by the following rule: the γ-factors shall be permuted in all possible ways with respect to their position among the L-operators. The indices of L and γ then decreases from left to right, one step for an entity preceded by an L_i, two steps for an entity preceded by an γ_j. The highest index to the left is $(k-1)$ in all terms. This rule completely determines the expression. Example:

$$A_{l-5}^{l,l'} = (L_4 L_3 L_2 L_1 L_0 + L_4 L_3 L_2 \gamma_1 + L_4 L_3 \gamma_2 L_0 + L_4 \gamma_3 L_1 L_0 + \gamma_4 L_2 L_1 L_0 + L_4 \gamma_3 \gamma_1 + \gamma_4 L_2 \gamma_1 + \gamma_4 \gamma_2 L_0)\, A_l^{l,l'}.$$

As a result we get $A^{ll'}_{l-k}(\alpha; l_0, \nu)$ expressed as a sum of known functions $D^{l+p,l'+\mu}_{l-p}$ where $p \leqslant k$, $p \geqslant \mu \geqslant \max(l_0 - l', -p)$ i.e.

$$A^{l,l'}_{l-k}(\alpha; l_0, \nu) = \sum_{p \leq k} \sum_{\mu=\max(l_0-l',-p)}^{\mu=p} \lambda_{p\mu} D^{l-p,l'+\mu}_{l-p}(y; l_0, \nu). \tag{5.3}$$

We remark that the terms on the r.h.s. of (5.3) may be rearranged in the following way: we first note that

$$D^{l-p,l'+\mu}_{l-p}\left(\frac{x+1}{x-1}; l_0, \nu\right) = \frac{(x^2-1)}{4} D^{l'-p-2,l'+\mu}_{l-p-2}\left(\frac{x+1}{x-1}; l_0, \nu\right),$$

i.e. from $A^{ll'}_n$ we can take out a factor $(x^2-1)^{\frac{1}{2}(n+1)}$ (cf. (3.7)). The rest will then contain Jacobi polynomials multiplied by different numbers of factors (x^2-1) which gives new Jacobi polynomials. If we put

$$v^n_k(x) = (x^2-1)^{\frac{1}{2}(n+1)}[(x-1)^{\alpha/2}(x+1)^{\beta/2}P^{\alpha,\beta}_k(x) - 2^{2l_0}(x-1)^{-\alpha/2}(x+1)^{-\beta/2}P^{-\alpha,-\beta}_{k+\alpha+\beta}(x)],$$

with $\alpha = l_0 - i\nu$, $\beta = l_0 + i\nu$, $A^{ll'}_n$ can be expressed in the following way

$$A^{l,l'}_n(\alpha; l_0, \nu) = \sum_k^{l'+l-l_0-n} a^n_k v^n_k(x),$$

We can now apply the ladder operators L_+ and L_- and in this way we arrive at a recursion formulae for the coefficients a^n_k containing five consecutive coefficients.

Using the variable $v = \operatorname{Tgh}\alpha$, (5.3) may be rewritten in the following way:

$$\begin{aligned} A^{l,l'}_n(\alpha; l_0, \nu) = {} & v^{-l'-l-1}(1-v)^{\frac{1}{2}(n+1+l_0-i\nu)}(1+v)^{\frac{1}{2}(n+1+l_0+i\nu)}P^{l,l'}_n(v; l_0, \nu) \\ & - v^{-l'-l-1}(1-v)^{\frac{1}{2}(n+1-l_0+i\nu)}(1+v)^{\frac{1}{2}(n+1-l_0-i\nu)}P^{l,l'}_n(v; -l_0, -\nu), \end{aligned} \tag{5.4}$$

where $P^{ll'}_n(v; l_0, \nu)$ is a polynomial of degree $l' + l - l_0 - n$:

$$P^{l,l'}_n(v; l_0, \nu) = \sum_{k=0}^{l'+l-l_0-n} p_k(n; l_0, \nu) v^k.$$

L_- and L_+ give recursion formulas for the coefficients p_k. These formulas are found in Appendix 3.

More explicitly (5.4) reads in the case of l_0 integer:

$$\begin{aligned} & A^{ll'}_n(\alpha; l_0, \nu) \\ & = N(l, l', n; l_0, \nu) \cdot v^{-l'-l-1} \cdot (1-v^2)^{\frac{1}{2}(n+1)} \cdot \{(1-v)^{\frac{1}{2}(l_0-i\nu)}(1+v)^{\frac{1}{2}(l_0+i\nu)}P'^{ll'}_n(v; l_0, \nu) \\ & \quad - (1-v)^{-\frac{1}{2}(l_0-i\nu)}(1+v)^{-\frac{1}{2}(l_0+i\nu)}P'^{ll'}_n(v, -l_0, -\nu)\}, \end{aligned} \tag{5.5}$$

where $$N(l, l', n; l_0, \nu) = \frac{(-1)^{l'+l_0-n} \cdot 2^{-l'-l-1}(l'+l)!}{i\nu}.$$

$$\sqrt{\frac{(2l'+1)!\,(2l+1)!\,(2l')!\,(2l)!}{(1+\nu^2)\dots(l'^2+\nu)\,(1+\nu^2)\dots(l^2+\nu^2)\,(l'+l_0)!\,(l'-l_0)!\,(l+l_0)!\,(l-l_0)!\,(l'+n)!\,(l'-n)!\,(l+n)!\,(l-n)!}},$$

and $P_n^{ll'}(v, l_0, \nu) = p_0(n, l_0, \nu)\, P_n'^{ll'}(v, l_0, \nu)$ i.e. the constant in $P_{n'}'^{ll'}(v, l_0\, \nu)$ is equal to one.

For l_0 half-integer we get a similar expression, the only difference being that the factor

$$\frac{(-1)^{l'+l_0-n}}{i\nu\sqrt{(1+\nu^2)\dots(l'^2+\nu^2)(1+\nu^2)\dots(l^2+\nu^2)}}$$

is replaced by
$$\frac{(-1)^{l'+l_0+\frac{1}{2}-n}}{\sqrt{(\frac{1}{4}+\nu^2)\dots(l'^2+\nu^2)(\frac{1}{4}+\nu^2)\dots(l^2+\nu^2)}}.$$

Here we still have the restriction $l \leqslant l'$ but $A_n^{ll'}(\alpha; l_0, \nu)$ with $l > l'$ can now be determined using (4.4) and (5.5).

II. $\nu = 0$, l_0 *half-integer*

In this case the operator $O(n)$ is

$$2n\left(\operatorname{Sh}\alpha\frac{d}{d\alpha} + \operatorname{Ch}\alpha\right) = 2n\left[(\operatorname{Ch}^2\alpha - 1)\frac{d}{d(\operatorname{Ch}\alpha)} + \operatorname{Ch}\alpha\right].$$

Using the properties of the Legendre-functions and equation (2.3) we now get $A_{l-1}^{ll'}(\alpha; l_0, 0)$ as a linear combination of $A_{l-1}^{l-1,l'+1}(\alpha; l_0, 0)$ and $A_{l-1}^{l-1,l'-1}(\alpha, l_0, 0)$. Repeated application of (2.3) then gives an expansion analogous to (5.3).

6. Concluding remarks

We have determined the functions $A_n^{ll'}(\alpha, l_0, \nu)$ without imposing any condition on their behaviour in the limit $\alpha \to \infty$ i.e. $v \to 1$. As we shall show below, we have

$$\lim_{\alpha\to\infty} A_n^{l,l'}(\alpha; l_0, \nu) = 0.$$

To see this we observe that the ladder operator K_+ does not affect the behaviour in the limit $v \to 1$. That this must be so is also clear from the form of the γ's, which do not depend on l. We can now restrict ourselves to the case $n \geqslant 0$. For $n \leqslant l_0$ the behaviour of $A_n^{ll'}$ as $v \to 1$ is the same as that of $A_n^{l_0 l'}(\alpha, l_0\, \nu)$ which for $v = 1$ has the exponent $\frac{1}{2}(l_0 + 1 - n + i\nu)$ (cf. (4.6)). In the same way we get that for $l_0 < n$ the behaviour of $A_n^{ll'}$ as $v \to 1$ is the same as that of $A_n^{nl'}(\alpha, l_0, \nu)$, i.e. it has in $v = 1$ the exponent $\frac{1}{2}(n + 1 - l_0 + i\nu)$. We thus have the property (6.1).

In the matrix elements $T_{mm'}^{ll'}$ there always occurs a term with $l_0 = n$ i.e. the matrix element as a whole has for $v = 1$ the exponent $\frac{1}{2}(1 + i\nu)$, which is the result given by Bargmann [6]. The $T_{mm'}^{ll'}$ are not square-integrable over the HLG. The integration in the variable α goes over the interval $(0, \infty)$ with the measure $\operatorname{Sh}^2\alpha\, d\alpha$ i.e. $(1 - v^2)^{-2}\, dv$ and thus this integral diverges. The complete orthogonality relation is given in ref. [4].

We conclude with a few remarks on the connection between the expressions derived above and the integral expressions for $A_n^{ll'}(\alpha; l_0, \nu)$ which one gets from the explicit realization of a representation space given in § 11 of ref. [1]. It turns out that in the case of $n=1$, $l \leqslant l'$ the integral obtained from ref. [1] is essentially Euler's integral formula for the hypergeometric function [7]. In the general case Euler's integral formula can be used to derive an expression for $A_n^{ll'}(\alpha; l_0, \nu)$ containing a double sum of hypergeometric functions of the variable $1-e^{-2\alpha}$. However, we will not go into the details of this here [8].

ACKNOWLEDGEMENT

The author wishes to express his sincere thanks to professor Nils Svartholm, who proposed this study, for his continued interest and valuable criticism.

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APPENDIX 1

Differential operator expressions for M_i, N_i, $\mathbf{M}^2-\mathbf{N}^2$ and $\mathbf{M}\cdot\mathbf{N}$

$$M_1 = i\left(-\frac{s_1\cdot c_2}{s_2}\frac{\partial}{\partial\varphi}+c_1\cdot\frac{\partial}{\partial\theta}+\frac{s_1}{s_2}\frac{\partial}{\partial\psi}\right),$$

$$M_2 = i\left(-\frac{c_1\cdot c_2}{s_2}\frac{\partial}{\partial\varphi}-s_1\frac{\partial}{\partial\theta}+\frac{c_1}{s_2}\cdot\frac{\partial}{\partial\psi}\right),$$

$$M_3 = i\frac{\partial}{\partial\varphi},$$

$$N_1 = i\left[\frac{c_1 C}{s_2 S}\frac{\partial}{\partial\varphi}+\frac{s_1 c_2 C}{S}\frac{\partial}{\partial\theta}-\left\{\frac{c_4(c_1c_3-s_1s_3c_2)}{s_4 S}+\frac{c_2c_1 C}{s_2 S}\right\}\frac{\partial}{\partial\psi}+s_1s_2\frac{\partial}{\partial\alpha}\right.$$
$$\left.-\frac{c_1s_3+s_1c_3c_2}{S}\frac{\partial}{\partial\beta}+\frac{c_1c_3-s_1s_3c_2}{s_4 S}\frac{\partial}{\partial\gamma}\right],$$

$$N_2 = i\left[-\frac{s_1 C}{s_2 S}\frac{\partial}{\partial\varphi}+\frac{c_1c_2 C}{S}\frac{\partial}{\partial\theta}+\left\{\frac{c_4(s_1c_3+c_1s_3c_2)}{s_4 S}+\frac{s_1c_2 C}{s_2 S}\frac{\partial}{\partial\psi}\right\}+c_1s_2\frac{\partial}{\partial\alpha}\right.$$
$$\left.+\frac{s_1s_3-c_1c_3c_2}{S}\frac{\partial}{\partial\beta}-\frac{s_1c_3+c_1s_3c_2}{s_4 S}\frac{\partial}{\partial\gamma}\right],$$

$$N_3 = i\left[-\frac{s_2 C}{S}\frac{\partial}{\partial\theta}-\frac{s_2s_3c_4}{s_4 S}\frac{\partial}{\partial\psi}+c_2\frac{\partial}{\partial\alpha}+\frac{s_2c_3}{S}\frac{\partial}{\partial\beta}+\frac{s_2s_3}{s_4 S}\frac{\partial}{\partial\gamma}\right],$$

$$
\begin{aligned}
\mathbf{M}^2-\mathbf{N}^2 = & \frac{1}{s_2^2 S^2}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{S^2}\frac{\partial^2}{\partial\theta^2}+\left[\frac{c_2^2}{s_2^2 S^2}+\frac{c_4^2}{s_4^2 S^2}-1+2\frac{c_2 c_3 c_4 C}{s_2 s_4 S^2}\right]\frac{\partial^2}{\partial\psi^2}+\frac{\partial^2}{\partial\alpha^2}+\frac{1}{S^2}\frac{\partial^2}{\partial\beta^2} \\
& +\frac{1}{s_4^2 S^2}\cdot\frac{\partial^2}{\partial\gamma^2}-2\left[\frac{c_2}{s_2^2 S^2}+\frac{c_3 c_4 C}{s_2 s_4 S^2}\right]\frac{\partial^2}{\partial\varphi\partial\psi}-2\frac{s_3 C}{s_2 S^2}\frac{\partial^2}{\partial\varphi\partial\beta}+2\frac{c_3 C}{s_2 s_4 S}\frac{\partial^2}{\partial\varphi\partial\gamma} \\
& +2\frac{s_3 c_4 C}{s_4 S^2}\frac{\partial^2}{\partial\theta\partial\psi}-2\frac{c_3 C}{S^2}\frac{\partial^2}{\partial\theta\partial\beta}-2\frac{s_3 C}{s_4 S^2}\frac{\partial^2}{\partial\theta\partial\gamma}+2\frac{c_2 s_3 C}{s_2 S^2}\frac{\partial^2}{\partial\psi\partial\beta} \\
& -2\left[\frac{c_4}{s_4^2 S^2}+\frac{c_2 c_3 C}{s_2 s_4 S^2}\right]\frac{\partial^2}{\partial\psi\partial\gamma}+\frac{c_2}{s_2 S^2}\frac{\partial}{\partial\theta}+2\frac{C}{S}\frac{\partial}{\partial\alpha}+\frac{c_4}{s_4 S^2}\frac{\partial}{\partial\beta},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}\cdot\mathbf{N} = & -\frac{c_2 c_4 s_3}{s_2 s_4 S}\frac{\partial^2}{\partial\psi^2}+\frac{s_3 c_4}{s_2 s_4 S}\frac{\partial^2}{\partial\varphi\partial\psi}-\frac{c_3}{s_2 S}\frac{\partial^2}{\partial\varphi\partial\beta}-\frac{s_3}{s_2 s_4 S}\frac{\partial^2}{\partial\varphi\partial\gamma}+\frac{c_3 c_4}{s_4 S}\frac{\partial^2}{\partial\theta\partial\psi}+\frac{s_3}{S}\frac{\partial^2}{\partial\theta\partial\beta} \\
& -\frac{c_3}{s_4 S}\frac{\partial^2}{\partial\theta\partial\gamma}-\frac{\partial^2}{\partial\psi\partial\alpha}+\frac{c_2 c_3}{s_2 S}\frac{\partial^2}{\partial\psi\partial\beta}+\frac{c_2 s_3}{s_2 s_4 S}\frac{\partial^2}{\partial\psi\partial\gamma}-\frac{C}{S}\frac{\partial}{\partial\psi},
\end{aligned}
$$

where $\quad c_1=\cos\varphi,\quad c_2=\cos\theta,\quad c_3=\cos\psi,\quad c_4=\cos\beta,$

$s_1=\sin\varphi,\quad s_2=\sin\theta,\quad s_3=\sin\psi,\quad s_4=\sin\beta,$

$C=\operatorname{Cosh}\alpha,\quad S=\operatorname{Sinh}\alpha.$

APPENDIX 2

The differential equation for $A_n^{ll}(\alpha; l_0, v)$

$$
\left\{v^4(1-v^2)^4\frac{d^4}{dv^4}+p(v)\,v^3(1-v^2)^3\frac{d^3}{dv^3}+q(v)\,v^2(1-v^2)^2\frac{d^2}{dv^2}+r(v)\,v(1-v^2)\frac{d}{dv}+s(v)\right\}
\times A_n^{l,l'}(\alpha; l_0, \nu)=0,
$$

where $v=\operatorname{Tgh}\alpha$ and

$$p(v)=-2(7v^2-3),$$

$$q(v)=50v^4-48v^2+6+2(l(l+1)+l'(l'+1)+1+\nu^2-l_0^2-n^2)\,v^2-2l(l+1)-2l'(l'+1),$$

$$
\begin{aligned}
r(v)= & 8v^2(5v^2-3)(1-v^2)-2(l(l+1)+l'(l'+1))(1-v^2) \\
& +6(l(l+1)+l'(l'+1)+1+\nu^2-l_0^2-n^2)\,v^2(1-v^2)-8nil_0\nu v,
\end{aligned}
$$

$$
\begin{aligned}
s(v)= & 2(l(l+1)+l'(l'+1)+1+\nu^2-l_0^2-n^2)\,v^6+(l(l+1)+l'(l'+1)+1+\nu^2-l_0^2-n^2) \\
& \times(l(l+1)+l'(l'+1)-3+\nu^2-l_0^2-n^2)\,v^4+[4\nu^2(n^2+l_0^2)-4n^2l_0^2 \\
& +2(l(l+1)+l'(l'+1)-1)(l_0^2+n^2)+2(l^2(l+1)^2+l'^2(l'+1)^2-1) \\
& -2(l(l+1)+l'(l'+1)-1)\,\nu^2]\,v^2+(l'-l)^2(l'+l+1)^2-4nil_0\nu v(1+v^2).
\end{aligned}
$$

APPENDIX 3

Recursion relations for $p_k(n, l_0, \nu)$

From (2.4) we get

$$\begin{aligned}-2\beta(n+1)\,p_k(n+1;l_0,\nu)\\ &= p_k(n;l_0,\nu)\,[d_0 k(k-1)+e_0 k+h_0]+p_{k-1}(n;l_0,\nu)\,[e_1(k-1)+h_1]\\ &\quad + p_{k-2}(n;l_0,\nu)\,[d_2(k-2)(k-3)+e_2(k-2)+h_2],\end{aligned} \tag{A 3.1}$$

where $d_0=1,\quad e_0=-2(l'+l-n),\quad h_0=2(l-n)(l'-n),$

$d_2=-1,\quad e_1=2i\nu,\quad h_1=-2i\nu(l'+l-l_0-n),$

$e_2=2(l'+l-n-l_0-1),\quad h_2=-(l'+l-l_0-n)(l'+l-n-1).$

(2.5) gives

$$\begin{aligned}-2\beta(n)\,p_k(n-1;l_0,\nu)\\ &= p_k(n;l_0,\nu)\,[a_0 k(k-1)+b_0 k+c_0]+p_{k-1}(n;l_0,\nu)\,[b_1(k-1)+c_1]\\ &\quad + p_{k-2}(n;l_0,\nu)\,[a_2(k-2)(k-3)+b_2(k-2)+c_2]\\ &\quad + p_{k-3}(n;l_0,\nu)\,[b_3(k-3)+c_3]\\ &\quad + p_{k-4}(n;l_0,\nu)\,[a_4(k-4)(k-5)+b_4(k-4)+c_4],\end{aligned} \tag{A 3.2}$$

where $a_0=1,\quad b_0=-2(l'+l+n),$

$a_2=-2,\quad b_1=2i\nu,$

$a_4=1,\quad b_2=2(2l'+2l-l_0-1),$

$b_3=-2i\nu,$

$b_4=-2(l'+l-l_0-n-1),$

$c_0=2(l+n)(l'+n),$

$c_1=-2i\nu(l'+l+l_0+n),$

$c_2=l(1-l)+l'(1-l')-(n-l_0)^2+2l_0(l'+l)+2n^2-n-l_0-4ll',$

$c_3=2i\nu(l'+l-l_0),$

$c_4=(l'+l-l_0-n)(l'+l-l_0-n-1).$

From (A 3.1) and (A 3.2) we get a recursion relation between seven consecutive coefficients $p_k(n;l_0,\nu)$. In particular (A 3.2) gives

$$-2\beta(n)\,p_0(n-1;l_0,\nu)=p_0(n;l_0,\nu)\,c_0=2(l+n)(l'+n)\,p_0(n;l_0,\nu)$$

and thus

$$p_0(n,l_0,\nu)=(-1)^{l-n}\sqrt{\frac{2l!\,(l'+l)!\,(l'-l)!}{(l+n)!\,(l-n)!\,(l'+n)!\,(l'-n)!}}\,p_0(l;l_0,\nu)=p_0(n;-l_0,-\nu).$$

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Tryckt den 29 juli 1965

Uppsala 1965. Almqvist & Wiksells Boktryckeri AB

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Band 33 nr 33

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A note on the matrix elements of a unitary representation on the homogeneous Lorentz group

ALMQVIST & WIKSELL
STOCKHOLM
1967

3.66 10 5 Communicated 1 June 1966 by N. SVARTHOLM

A note on the matrix elements of a unitary representation of the homogeneous Lorentz group

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ABSTRACT

In the present paper the integral expression for the matrix elements of a finite Lorentz transformation in a unitary representation of the homogeneous Lorentz group is studied. In particular the connection between the expressions obtained in this way and those obtained by infinitesimal methods is investigated.

1. Introduction

In a previous paper [1] we have studied the matrix elements of a finite Lorentz transformation in a unitary representation of the homogeneous group $L(1, 3)$. The matrix elements were computed in an angular momentum basis. The infinitesimal method was employed and the results could be expressed in terms of hypergeometric functions. In the present paper we will consider the explicit realization of a unitary representation of $L(1, 3)$ belonging to the principal series given in § 11 of reference [2]. We then get an expression for the matrix elements in the form of an integral which in principle contains all information about them. However, the functional dependence on the physical parameters is not very clear and it is therefore of interest to perform the integration. The results of reference [1] will then serve as guide how this should be done. For a special case we show explicitly how to obtain the same expression as that derived in [1]. The integration can be performed also in the general case and in this way a new expression for the general matrix elements is obtained.

In section 2 the relevant facts concerning the realization of the representations described in [2] are given and in section 3 and 4 the integration will be performed. The results obtained in this report were mentioned at the end of reference [1]. The notations will be the same as those in [1].

2. A realization of the unitary representations (l_0, ν) of $L(1, 3)$

According to § 11 of ref. [2] an explicit realization of a basis

$$\{|l, m; l_0, \nu\rangle\}, \quad l = l_0, l_0 + 1, \ldots; \quad m = -l, -l+1, \ldots, l-1, l,$$

in which $\mathbf{M}^2$ and M_3 are diagonal is

$$|l, m; l_0, \nu\rangle = N(l, l_0, \nu)\, D^l_{l_0, m}(\varphi_1, \theta_1, \psi_1), \tag{2.1}$$

where
$$N(l, l_0, \nu) = \omega(l, l_0)\frac{\sqrt{2l+1}}{4\pi}\sum_{k=l_0}^{k=l}\frac{(-k+i\nu)}{\sqrt{k^2+\nu^2}}, \quad |\omega(l, l_0)| = 1. \tag{2.2}$$

In reference [2] the general expression for the operator representing an arbitrary Lorentz transformation is given. Since we consider the following parametrization of an arbitrary Lorentz transformation L:

$$L = R_z(\varphi)\, R_x(\theta)\, R_z(\psi)\, A_z(\alpha)\, R_x(\beta)\, R_z(\gamma) \tag{2.3}$$

(cf. [1]), we only need the explicit expression for the operator V_α which corresponds to an acceleration in the z-direction with the velocity $v = \operatorname{Tgh} \alpha$. The action of V_α is defined by

$$V_\alpha D^l_{l_0, m}(\varphi_1, \theta_1, \psi_1) = (\operatorname{Ch}\alpha - \cos\theta_1 \operatorname{Sh}\alpha)^{i\nu-1} D^l_{l_0, m}(\varphi_1\, \theta_1', \psi_1), \tag{2.4}$$

where
$$\cos\theta_1' = \frac{\cos\theta_1 - \operatorname{Tgh}\alpha}{1 - \cos\theta_1 \operatorname{Tgh}\alpha}. \tag{2.5}$$

In the basis $\{|l, m; l_0\, \nu\rangle\}$ the matrix elements of the operator corresponding to an arbitrary L is

$$T^{ll'}_{mm'}(\varphi, \theta, \psi, \alpha, \beta, \gamma; l_0, \nu) = \sum_{n=-\min(l,l')}^{n=\min(l,l')} D^l_{m,n}(\varphi, \theta, \psi)\, A^{ll'}_n(\alpha; l_0\, \nu)\, D^{l'}_{n, m'}(0, \beta, \gamma). \tag{2.6}$$

The functions $A^{ll'}_n(\alpha; l_0, \nu)$ are the matrix elements of the operator V_α:

$$A^{ll'}_n(\alpha; l_0, \nu) = \langle l, n; l_0\, \nu | V_\alpha | l'\, n; l_0, \nu\rangle. \tag{2.7}$$

More explicitly this equation reads

$$A^{ll'}_n(\alpha; l_0, \nu) = 4\pi^2\, \overline{N(l, l_0, \nu)}\, N(l', l_0, \nu)$$
$$\times \int_0^\pi P^l_{l_0, n}(\cos\theta_1)\, \{\operatorname{Ch}\alpha - \cos\theta_1 \operatorname{Sh}\alpha\}^{i\nu-1} P^{l'}_{l_0, n}(\cos\theta_1') \sin\theta_1\, d\theta_1 \tag{2.8}$$

where the bar denotes complex conjugation and where we have used the notation

$$D^l_{m,n}(\varphi_1, \theta_1, \psi_1) = e^{-im\varphi_1} P^l_{m,n}(\cos\theta_1)\, e^{-in\psi_1}.$$

The formula (2.8) will be the starting point for the derivations of the next two sections.

3. Integration of the expression for $A^{ll_0}_n(\alpha; l_0, \nu)$

From the results of reference [1] it follows that $A^{ll_0}_n(\alpha; l_0, \nu)$ can be expressed in the following way:

$$A_n^{ll_0}(\alpha; l_0, \nu) = \text{const. } (\text{Sh } \alpha)^{l-l_0} e^{\alpha(i\nu - l - n - 1)} F(l+n+1, l+1-i\nu, 2l+2, 1-e^{-2\alpha}). \quad (3.1)$$

We will show how this expression for $A_n^{ll_0}(\alpha; l_0, \nu)$ can be obtained from (2.8). The following explicit expressions for $P^l_{m,n}(\mu)$ will be used [3]:

$$P^l_{m,n}(\mu) = A(l, m, n)\,(1-\mu)^{-(n-m)/2}(1+\mu)^{-(n+m)/2}\frac{d^{l-n}}{d\mu^{l-n}}\{(1-\mu)^{l-m}(1+\mu)^{l+m}\} \quad (3.2\,\text{a})$$

$$= A(l, n, m)\,(1-\mu)^{-(m-n)/2}(1+\mu)^{-(m+n)/2}\frac{d^{l-m}}{d\mu^{l-m}}\{(1-\mu)^{l-n}(1+\mu)^{l+n}\}, \quad (3.2\,\text{b})$$

where
$$A(l, m, n) = \frac{(-1)^{l-m}\, i^{n-m}}{2^l(l-m)!}\sqrt{\frac{(l-m)!\,(l+n)!}{(l+m)!\,(l-n)!}}. \quad (3.3)$$

With $\mu = \cos\theta_1$, $\mu' = \cos\theta_1'$ it follows from (3.2 b) that

$$P^{l_0}_{l_0,n}(\mu') = A(l_0, n, l_0)\,(1-\mu')^{(l_0-n)/2}(1+\mu')^{(l_0+n)/2}$$
$$= A(l_0, n, l_0)\,(1-\mu)^{(l_0-n)/2}(1+\mu)^{(l_0+n)/2} e^{-\alpha n}(\text{Ch }\alpha - \mu\text{ Sh }\alpha)^{-l_0}.$$

Inserting (3.2 b) for $P^l_{l_0,n}(\mu)$ into (2.8) one obtains

$$A_n^{ll_0}(\alpha; l_0, \nu) = N_1 e^{-\alpha n}\int_{-1}^{1}[\text{Ch }\alpha - \mu\text{ Sh }\alpha]^{-l_0-1+i\nu}\frac{d^{l-l_0}}{d\mu^{l-l_0}}\{(1-\mu)^{l-n}(1+\mu)^{l+n}\}\,d\mu$$

$$= N_1(-1)^{l-l_0}\frac{\Gamma(l+1-i\nu)}{\Gamma(l_0+1-i\nu)}(\text{Sh }\alpha)^{l-l_0}e^{-\alpha n}$$

$$\times\int_{-1}^{+1}(1-\mu)^{l-n}(1+\mu)^{l+n}[\text{Ch }\alpha - \mu\text{ Sh }\alpha]^{i\nu-l-1}\,d\mu, \quad (3.4)$$

where
$$N_1 = 4\mu^2\,\overline{N(l, l_0, \nu)}\,N(l_0, l_0, \nu)\,\overline{A(l, n, l_0)}\,A(l_0, n, l_0).$$

With $\mu + 1 = 2t$ inserted into (3.4) we get

$$A_n^{ll_0}(\alpha; l_0, \nu) = N_1(-1)^{l-l_0}\frac{\Gamma(l+1-i\nu)}{\Gamma(l_0+1-i\nu)}2^{2l+1}(\text{Sh }\alpha)^{l-l_0}e^{\alpha(i\nu-l-1-n)}$$

$$\times\int_0^1 t^{l+n}(1-t)^{l-n}(1-t(1-e^{-2\alpha}))^{-l-1+i\nu}\,dt. \quad (3.5)$$

From Euler's integral formula for the hypergeometric function [4]

$$F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \quad (3.6)$$

it now follows that we have the expression (3.1) for $A_n^{ll_0}(\alpha; l_0, \nu)$. The value of the constant is given by (3.5) and (3.6).

4. Integration of the expression for $A_n^{ll'}(\alpha; l_0, \nu)$

In reference [1] the functions $A_n^{ll'}(\alpha; l_0, \nu)$ were determined by means of recursion relations from $A_l^{ll'}(\alpha; l_0, \nu)$ which can be given in closed form. However, no general formula was given for the coeficients in the resulting expression. In this section we will show how an explicit summation formula can be obtained from (2.8). The final step in the derivation once more employs Euler's integral formula (3.6) for the hypergeometric function. We use (3.2 a) for $P^l_{l_0,n}(\mu')$ and write

$$P^l_{l_0,n}(\mu) = A(l, l_0, n)\,(1-\mu)^{-(n-l_0)/2}(1+\mu)^{-(n+l_0)/2}$$
$$\times \sum_{k=0}^{l-n} \binom{l-n}{k} (-1)^k \frac{(l-l_0)!\,(l+l_0)!}{(l-l_0-k)!\,(l_0+n+k)!}\,(1-\mu)^{l-l_0-k}(1+\mu)^{l_0+n+k}, \qquad (4.1)$$

and express $P^{l'}_{l_0,n}(\mu')$ similarly. Using

$$(1+\mu')(1-\mu')^{-1} = (1+\mu)(1-\mu)^{-1}e^{-2\alpha},\ (1-\mu')(1+\mu')$$
$$= (1-\mu)(1+\mu)(\operatorname{Ch}\alpha - \mu \operatorname{Sh}\alpha)^{-2},\ (1-\mu') = e^{\alpha}(\operatorname{Ch}\alpha - \mu\operatorname{Sh}\alpha)^{-1}(1-\mu),$$

it is easy to see that (2.8) gives

$$A_n^{ll'}(\alpha; l_0, \nu) = N_2\, e^{-\alpha(l_0+n+1-i\nu)} \sum_{k=0}^{l-n}\sum_{k'=0}^{l'-n}(-1)^{k+k'}\binom{l-n}{k}\binom{l'-n}{k'} e^{-2\alpha k'}$$
$$\times \frac{(l'-l_0)!\,(l'+l_0)!\,(l-l_0)!\,(l+l_0)!}{(l'-l_0-k')!\,(l_0+n+k')!\,(l-l_0-k)!\,(l_0+n+k)!}\int_{-1}^{+1}(1-\mu)^{l+l'-l_0-n-(k+k')}$$
$$\times (1+\mu)^{l_0+n+k+k'}\left(1-\frac{\mu+1}{2}(1-e^{-2\alpha})\right)^{-l'-1+i\nu} d\mu, \qquad (4.2)$$

where $$N_2 = 4\pi^2\,\overline{N(l, l_0, \nu)}\,N(l', l_0, \nu)\,A(l, l_0, n)\,A(l', l_0, n).$$

In (4.1) and all the following expressions we use the factorial notation but whenever necessary it should be interpreted as denoting the Γ-function. Thereby the summations are restricted appropriately.

Introduce once more $\mu + 1 = 2t$. The integral in (4.2) is then

$$2^{l+l'+1}\int_0^1 (1-t)^{l+l'-l_0-n-(k+k')}\,t^{l_0+n+k+k'}\,(1-t(1-e^{-2\alpha}))^{-l'-1+i\nu}\,dt$$
$$= 2^{l+l'+2}\frac{(l+l'-l_0-n-(k+k'))!\,(l_0+n+k+k')!}{(l+l'+1)!}$$
$$\times F(l'+1-i\nu, l_0+n+k+k'+1, l+l'+2, 1-e^{-2\alpha}).$$

After a change to the summation variables $p = k+k'$ and k' the complete expression for $A_n^{ll'}(\alpha; l_0, \nu)$ reads

$$A_n^{ll'}(\alpha; l_0, \nu) = N_3\, e^{-\alpha(l_0+n+1-i\nu)} \sum_p n_p(\alpha)\,(-1)^p\,(l_0+n+p)!\,(l+l'-l_0-n-p)!$$
$$\times F(l'+1-i\nu, l_0+n+p+1, l+l'+2, 1-e^{-2\alpha}),$$

where

$$N_3 = N_2 \cdot 2^{l+l'+1}\,\frac{(l-n)!\,(l'-n)!\,(l-l_0)!\,(l'-l_0)!\,(l+l_0)!\,(l'+l_0)!}{(l+l'+1)!}$$

and

$$n_p(\alpha) = \sum_{k'} \frac{e^{-2\alpha k'}}{k'!\,(l'-n-k')!\,(l_0+n+p-k')!\,(p-k')!\,(l'-l_0-k')!} \cdot \frac{}{\times (l-n-p+k')!\,(l_0+n+k')!\,(l-l_0-p+k')!}$$

We note that if $n+l_0 \geqslant 0$ then the summation over k' starts at $k'=0$ and in that case $n_p(\alpha)$ can be expressed as a generalized hypergeometric function ${}_4F_3$ [4] in the following way:

$$n_p(\alpha) = \{(l'-n)!\,p!\,(l_0+n+p)!\,(l'-l_0)!\,(l-n-p)!\,(l_0+n)!\,(l-l_0-p)!\}^{-1}$$
$$\times {}_4F_3(n-l', -p, l_0-l', -l_0-n-p; l-n-p+1, l_0+n+1, l-l_0-p+1, e^{-2\alpha}).$$

$n_p(\alpha)$ can be related to a ${}_4F_3$-function also when $l_0+n<0$. In this case we introduce the new variable $k''=k'+l_0+n$. The summation over k'' then starts at $k''=0$. Further it is convenient to introduce $p''=p+l_0+n$. Thus $p-k'=p''-k''$. We then get

$$n_p(\alpha) = e^{2\alpha(l_0+n)}\{(l'+l_0)!\,(l_0+n+p'')!\,p''!\,(l'+n)!\,(l-n-p'')!\,(-l_0-n)!$$
$$(l-l_0-p'')!\}^{-1}{}_4F_3(-l'-l_0, -l_0-n-p'', -p'', -l'-n;$$
$$l-n-p''+1, -l_0-n+1, l-l_0-p''+1, e^{-2\alpha}).$$

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Tryckt den 18 januari 1967

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$$A_n^{ll'}(\alpha; l_0, \nu) = N_2 \, e^{[illegible]} \sum_p a_p(\alpha)(-1)^p (l_0+n+p)!\,(l+l'-l_0-n-p)!$$

[illegible]

where

$$N_2 = [illegible]$$

and

$$a_p(x) = \sum_{k'} [illegible]$$

We note that if $n + l_0 \geq 0$ then the summation over k' starts at $k' = 0$ and in that case $a_p(x)$ can be expressed as a generalized hypergeometric function ${}_4F_3$ [4] in the following way:

$$a_p(x) = [illegible]$$

$a_p(x)$ can be related to a ${}_4F_3$-function also when $l_0 + n < 0$. In this case we introduce the new variable $k'' = k' + l_0 + n$. The summation over k'' then starts at $k'' = 0$. Further it is convenient to introduce $p'' = p + l_0 + n$. Thus $p - k' = p'' - k''$. We then get

$$a_p(x) = [illegible] \tag{2.4}$$

where $N_2 = [illegible]$

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Tryckt den 16 januari 1967

Uppsala 1967. Almqvist & Wiksells Boktryckeri AB

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AWE

ALMQVIST & WIKSELL

STOCKHOLM

GÖTEBORG · UPPSALA

1965

ARKIV FÖR FYSIK Band 30 nr 21

3.65 08 2 Communicated 12 May 1965 by L. HULTHÉN and N. SVARTHOLM

On the contraction of representations of the Lorentz group to representations of the Euclidean group

By STAFFAN STRÖM

ABSTRACT

If the Lie algebra of the homogeneous Lorentz group $L(1, 3)$ is contracted with respect to the Lie algebra of the rotation group the resulting algebra is the Lie algebra of the threedimensional Euclidean group $E(3)$. In the present work we study the corresponding process of contraction in the unitary representations of the two groups. Since there is no general theory for contraction of representations, we use explicit expressions for the matrix elements of the unitary representations of $L(1, 3)$ and of $E(3)$ respectively. The latter are derived in the present paper while the former are known from earlier works. Finally explicit realizations, suggested by the contraction, of the unitary representations of $E(3)$ are given.

1. Introduction

If a Lie group, and in particular a simple or semi-simple Lie group, is contracted, in the sense of Inönü and Wigner [1], one obtains a new group which has an abelian invariant subgroup, i.e. it is non-semisimple. As the representation theory of non-semisimple Lie groups is less complete than that of simple Lie groups, it is of interest to see whether one can obtain representations of a non-semisimple Lie group by contraction of representations of a simple Lie group. Possible procedures for obtaining representations by the process of contraction, combined with special prescriptions, have been given by Inönü and Wigner [1]. There is, however, no general theory for the contraction of representations. In particular one does not know if the representations obtained by contraction form the complete set of representations of the group in question. However, in special cases this can be shown to be the case.

In the present work we show how the unitary representations of the three-dimensional Euclidean group $E(3)$ can be obtained by contraction of unitary representations of the homogeneous Lorentz group $L(1,3)$. The problem is first studied in the Lie algebras and it is then shown that the contraction can be performed also in the matrix elements of a finite group element. The matrix elements of the operator corresponding to a finite group element in a representation can in general be determined as a function of the parameters of the group even if one does not know explicitly the form of the operator and the Hilbert space on which this operator acts. This is a consequence of the fact that the matrix representation of the operator gives at the same time a linear representation of the group. An arbitrary column in such a matrix representation of the operator can be considered as a representation space. This space, however, is in general no Hilbert space. The fact that the matrix elements in the unitary representations of $L(1,3)$ and $E(3)$ are connected by the process of contraction can also be deduced from the fact that the contraction can be introduced into the explicit realization of the unitary representations of $L(1,3)$ given by Neumark [2]. We show how in this way one arrives at an explicit realization of the unitary representations of $E(3)$. This realization is very similar to the one obtained by an application of Mackey's method [3].

The representation theory of $E(3)$ is of course of interest in itself, but there are also applications of the results derived in this paper to physical problems due to the fact that the representations of the inhomogeneous Lorentz group can be expressed as direct integrals of representations of $E(3)$. We will return to these questions in a later paper.

In section 2 we review some well-known results concerning the representations of the Lie algebras of $L(1,3)$ and $E(3)$. It is shown that they are connected by a process of contraction. In section 3 we determine the form of the matrix element of a finite group element in $E(3)$ and in section 4 the results are compared to the corresponding ones for $L(1,3)$ which have been derived in ref. [4]. Finally we consider in section 5 a choice of realization of the representations of $E(3)$ which is suggested by the contraction.

2. The representations of the Lie algebras of $L(3,1)$ and $E(3)$

The Lie algebras of $L(1,3)$ and $E(3)$ can be described by the six hermitian basis elements $\mathbf{M}=(M_1, M_2, M_3)$, $\mathbf{N}=(N_1, N_2, N_3)$ and $\mathbf{M}=(M_1, M_2, M_3)$, $\mathbf{p}=(p_1, p_2, p_3)$ respectively. The commutation relations satisfied by these operators are

$$L(1,3): \qquad [M_r, M_s] = iM_t, \quad [M_r, N_r] = 0, \quad [N_r, N_s] = -iM_t,$$

$$[M_r, N_s] = [N_r, M_s] = iN_t, \quad (r, s, t \text{ in cyclic perm.}). \tag{2.1}$$

$$E(3): \qquad [M_r, M_s] = iM_t, \quad [M_r, p_r] = 0, \quad [p_r, p_s] = 0,$$

$$[M_r, p_s] = [p_r, M_s] = ip_t, \quad (r, s, t \text{ in cyclic perm.}). \tag{2.2}$$

We note that the commutation relations between M_r and p_s are the same as those between M_r and N_s. The only difference between the algebras is that the p_r:s commute with each other, whereas the N_r:s do not.

If the Lie algebra of $L(1,3)$ is contracted with respect to the subalgebra formed by M_1, M_2 and M_3, the resulting algebra is the Lie algebra of $E(3)$. The contraction can be described in the following way. Consider the algebra consisting of the elements $\mathbf{M}$, $\mathbf{K}=\lambda\mathbf{N}$. The commutation relations between M_r and K_t are the same as those between M_r and N_t, while

$$[K_r, K_s] = -i\lambda^2 \cdot M_t, \quad (r, s, t \text{ in cyclic perm.}),$$

i.e. we have formally $\qquad \lim_{\lambda\to 0} [K_r, K_s] = 0.$

In the limit of $\lambda \to 0$, $\mathbf{M}$ and $\mathbf{K}$ thus form an algebra which is isomorphic to the Lie algebra of $E(3)$.

The two invariants of $L(1,3)$ and $E(3)$ can be chosen as $\mathbf{M}^2-\mathbf{N}^2$, $\mathbf{M}\cdot\mathbf{N}$ and $\mathbf{p}^2$, $\mathbf{p}\cdot\mathbf{M}$ respectively. In a unitary irreducible representation of $L(1,3)$ belonging to the principal series [2] the values of the invariants are

$$\mathbf{M}^2-\mathbf{N}^2=l_0^2-1-\nu^2,$$

$$\mathbf{M}\cdot\mathbf{N}=l_0\cdot\nu, \tag{2.3}$$

where $l_0=\frac{1}{2}, 1, \frac{3}{2}, \ldots, -\infty<\nu<\infty$ or $l_0=0$, $0\leqslant\nu<\infty$. In the supplementary series the values are

$$\mathbf{M}^2-\mathbf{N}^2=c^2-1,$$

$$\mathbf{M}\cdot\mathbf{N}=0, \tag{2.4}$$

where $0\leqslant c\leqslant 1$. In a unitary irreducible representation of $E(3)$ one has

$$\mathbf{p}^2=p^2,$$

$$\mathbf{p}\cdot\mathbf{M}=l_0\cdot p, \tag{2.5}$$

where $p^2>0$, $l_0=0, \frac{1}{2}, 1, \ldots, p>0$ or $p<0$. If the operators K_i are introduced into equation (2.3) we get

$$\mathbf{K}^2=\lambda^2(\nu^2+1-l_0^2+\mathbf{M}^2),$$

$$\mathbf{K}\cdot\mathbf{M}=\lambda l_0 \nu. \tag{2.6}$$

Consider the following limit:

$$\lambda\to 0,\ |\nu|\to\infty,\ \text{so that}\ \lambda\nu=p. \tag{2.7}$$

We assume that λ is a positive parameter i.e. ν and p will have the same sign. Taking the limit (2.7) in the equations (2.6) one has formally

$$\lim_{(2.7)}\mathbf{K}^2=p^2,\quad \lim_{(2.7)}\mathbf{K}\cdot\mathbf{M}=l_0 p. \tag{2.8}$$

Since the parameter c in (2.4) is bounded, no such limit can be constructed for the supplementary series. This may be compared with the fact that there is no supplementary series of unitary representations of $E(3)$.

The relations (2.8) suggest that we investigate the corresponding limit in the representations. The basis vectors in the representation space of a unitary representation (l_0,ν) of $L(1,3)$ will be denoted by $|l, m; l_0, \nu\rangle$ and we assume that $\mathbf{M}^2$ and M_3 are diagonal in this basis with eigenvalues $l(l+1)$ and m respectively. Then $l=l_0, l_0+1\ldots$. According to ref. [2] a representation of the Lie algebra of $L(1,3)$ is then given by

$$M_+|l,m;l_0,\nu\rangle = \alpha^l_{m,m+1}|l,m+1;l_0,\nu\rangle,$$
$$M_-|l,m;l_0,\nu\rangle = \alpha^l_{m-1,m}|l,m-1;l_0,\nu\rangle,$$
$$M_3|l,m;l_0,\nu\rangle = m\,|l,m;l_0,\nu\rangle,$$
$$N_+|l,m;l_0,\nu\rangle = i\,\alpha^{l+1}_{-m,m+1}\gamma_1(l+1;l_0,\nu)\,|l+1,m+1;l_0,\nu\rangle$$
$$+\frac{\nu l_0}{l(l+1)}\,\alpha^l_{m,m+1}|l,m+1;l_0,\nu\rangle + i\,\alpha^l_{m+1,-m}\gamma_1(l;l_0,\nu)\,|l-1,m+1;l_0,\nu\rangle,$$
$$N_-|l,m;l_0,\nu\rangle = -i\,\alpha^{l+1}_{m-1,-m}\gamma_1(l+1;l_0,\nu)\,|l+1;m-1;l_0,\nu\rangle$$
$$+\frac{\nu l_0}{l(l+1)}\,\alpha^l_{m-1,m}|l,m-1;l_0,\nu\rangle - i\,\alpha^l_{-m,m-1}\gamma_1(l;l_0,\nu)\,|l-1,m-1;l_0,\nu\rangle,$$
$$N_3|l,m;l_0,\nu\rangle = -i\,\alpha^{l+1}_{m,m}\gamma_1(l+1;l_0,\nu)\,|l+1,m;l_0,\nu\rangle$$
$$+\frac{ml_0\nu}{l(l+1)}\,|l,m;l_0,\nu\rangle + i\,\alpha^l_{m,m}\gamma_1(l;l_0,\nu)\,|l-1,m;l_0,\nu\rangle,$$

where
$$M_\pm = M_1 \pm iM_2, \qquad N_\pm = N_1 \pm iN_2,$$

$$\gamma_1(l;l_0,\nu) = \frac{1}{l}\sqrt{\frac{(l^2-l_0^2)\,(l^2+\nu^2)}{(2l+1)\,(2l-1)}}, \quad \alpha^l_{m,n} = \sqrt{(l-m)\,(l+n)}. \tag{2.9}$$

A corresponding basis for a unitary representation (l_0, p) of $E(3)$ is denoted by $|l,m;l_0,p\rangle$. According to ref. [5] a representation of the Lie algebra of $E(3)$ is then given by

$$p_+|l,m;l_0,p\rangle = i\,\alpha^{l+1}_{-m,m+1}\gamma_2(l+1;l_0,\nu)\,|l+1,m+1;l_0,p\rangle$$
$$+\frac{pl_0}{l(l+1)}\,\alpha^l_{m,m+1}|l,m+1;l_0,p\rangle + i\,\alpha^l_{m+1,-m}\gamma_2(l;l_0,p)\,|l-1,m+1;l_0,p\rangle,$$
$$p_-|l,m;l_0,p\rangle = -i\,\alpha^{l+1}_{m-1,-m}\gamma_2(l+1;l_0,p)\,|l+1,m-1;l_0,p\rangle$$
$$+\frac{pl_0}{l(l+1)}\,\alpha^l_{m-1,m}|l,m-1;l_0,p\rangle - i\,\alpha^l_{-m,m-1}\gamma_2(l;l_0,p)\,|l-1,m-1;l_0,p\rangle,$$
$$p_3|l,m;l_0,p\rangle = -i\,\alpha^{l+1}_{m,m}\gamma_2(l+1;l_0,p)\,|l+1,m;l_0,p\rangle$$
$$+\frac{ml_0p}{l(l+1)}\,|l,m;l_0,p\rangle + i\,\alpha^l_{m,m}\gamma_2(l;l_0,p)\,|l-1,m;l_0,p\rangle,$$

where
$$p_\pm = p_1 \pm ip_2,$$
$$\gamma_2(l;l_0,p) = \frac{|p|}{l}\sqrt{\frac{(l^2-l_0^2)}{(2l+1)\,(2l-1)}}. \tag{2.10}$$

We have then made a choice of phases which is similar to the one in (2.9). $M_\pm$ and M_3 act upon $|l, m; l_0, p\rangle$ in the same way as upon $|l, m; l_0, \nu\rangle$. We note that

$$\lim_{(2.7)} \lambda\gamma_1(l; l_0, \nu) = \gamma_2(l; l_0, p), \quad \lim_{(2.7)} \lambda l_0 \nu = l_0 p. \tag{2.11}$$

Consequently the representations (l_0, ν) of the Lie algebra of $L(1,3)$ can be contracted, in the sense of the limit (2.7), to representations (l_0, p) of the Lie algebra of $E(3)$.

3. The matrix elements of a finite group element of $E(3)$ in a representation (l_0, p)

An arbitrary group element E of $E(3)$ can be expressed as a function of six parameters $(\varphi, \theta, \psi, a, \beta, \gamma)$ according to the factorization

$$E = R_z(\varphi)\, R_x(\theta)\, R_z(\psi)\, T_z(a)\, R_x(\beta)\, R_z(\gamma), \tag{3.1}$$

where e.g. $R_z(\varphi)$ is a rotation of an angle φ around the z-axis etc. and $T_z(a)$ is a translation of length a along the z axis. The matrix elements in a basis $|l, m; l_0, p\rangle$ of the operator corresponding to this element can then be written as follows

$$T^{ll'}_{mm'}(\varphi, \theta, \psi, a, \beta, \gamma; l_0, p) = \sum_{n=-\min(l,l')}^{n=\min(l,l')} D^l_{m,n}(\varphi, \theta, \psi)\, T^{ll'}_n(a; l_0, p)\, D^{l'}_{n,m'}(0, \beta, \gamma), \tag{3.2}$$

where $l \geqslant l_0 \geqslant 0$, $l' \geqslant l_0 \geqslant 0$. $D^l_{m,n}$ are the known matrix elements of a $(21+1)$-dimensional representation of the threedimensional rotation group, and $T^{ll'}_n(a; l_0, p)$ are new functions which are to be determined.

$T^{ll'}_n(a; l_0, p)$ will be determined by an application of the infinitesimal method. Having expressed an arbitrary element E of $E(3)$ as a function of the parameters $(\varphi, \theta, \psi, a, \beta, \gamma)$, the operators M_i and p_i of the Lie algebra can be written as partial differential operators in these variables. The explicit expressions for $p_\pm$, p_3, $\mathbf{p}^2$ and $\mathbf{p}\cdot\mathbf{M}$ are given in an appendix. The complete matrix elements $T^{ll'}_{mm'}$ for fixed l' and m' will satisfy the relations (2.10) and as a consequence satisfy the equations

$$\mathbf{p}^2\, T^{ll'}_{mm'} = p^2\, T^{ll'}_{mm'},$$

$$\mathbf{p}\cdot\mathbf{M}\, T^{ll'}_{mm'} = l_0 \cdot p\, T^{ll'}_{mm'}. \tag{3.3}$$

Inserting the expression (3.2) for $T^{ll'}_{mm'}$ into (2.10) and (3.3) and integrating over all angles φ, θ, ψ, β, γ one obtains relations containing only the functions $T^{ll'}_n(a; l_0, p)$. From (2.10) one gets three independent relations containing first order derivatives with respect to the variable a and differences in both indices l and n. From (3.3) one new relation is obtained, containing a second order derivative with respect to a and differences only in the index n. The abovementioned relations can be arranged to form ladder operators in the indices l and n, i.e. operators which increase or decrease one of these indices by one unit. Finally we may remark that the two equations (3.3) determine completely the form of the functions $T_n^{ll'}(a; l_0, p)$ apart from a common normalization factor.

The explicit form of the relations obtained from (2.10) is

$$\left[2na\frac{d}{da}+2n+2il_0pa\right]T_n^{ll'}(a;l_0,p)$$
$$=\alpha^l_{n-1,n}\alpha^{l'}_{n-1,n}T^{ll'}_{n-1}(a;l_0,p)-\alpha^l_{n,n+1}\alpha^{l'}_{n,n+1}T^{ll'}_{n+1}(a;l_0,p). \tag{3.4}$$

$$\alpha^{l+1}_{n,n}\left[a\frac{d}{da}-l\right]T_n^{ll'}(a;l_0,p)=-a(l+1)(2l+1)\gamma_2(l+1;l_0,p)\,T_n^{l+1,l'}(a;l_0,p)$$
$$-\tfrac{1}{2}\left[\alpha^{l+1}_{n+1,-n}\alpha^{l'+1}_{n+1,n}T^{ll'}_{n+1}(a;l_0,p)+\alpha^{l+1}_{-n,n-1}\alpha^{l'+1}_{n,n-1}T^{ll'}_{n-1}(a;l_0,p)\right]. \tag{3.5}$$

$$\alpha^{l}_{n,n}\left[a\frac{d}{da}+(l+1)\right]T_n^{ll'}(a;l_0,p)=al(2l+1)\gamma_2(l;l_0,p)\,T_n^{l-1,l'}(a;l_0,p)$$
$$+\tfrac{1}{2}\left[\alpha^{l}_{-n,n+1}\alpha^{l'}_{n,n+1}T^{ll'}_{n+1}(a;l_0,p)+\alpha^{l}_{n-1,-n}\alpha^{l'}_{n-1,n}T^{ll'}_{n-1}(a;l_0,p)\right]. \tag{3.6}$$

We define ladder operators $L_\pm$ and $K_\pm$ by

$$L_+(n)=a^2\frac{d^2}{da^2}+2a(n+1)\frac{d}{da}+2n(n+1)-l(l+1)-l'(l'+1)+a^2p^2+2il_0pa,$$

$$L_-(n)=a^2\frac{d^2}{da^2}-2a(n-1)\frac{d}{da}+2n(n-1)-l(l+1)-l'(l'+1)-a^2p^2-2il_0pa,$$

$$K_+(l)=a^2\frac{d^2}{da^2}-2la\frac{d}{da}+(l-l')(l+l'+1)+a^2p^2-\frac{2nil_0pa}{(l+1)},$$

$$K_-(l)=a^2\frac{d^2}{da^2}+2(l+1)a\frac{d}{da}+(l-l')(l+l'+1)+a^2p^2+\frac{2nil_0pa}{(l+1)}. \tag{3.7}$$

In terms of these operators $T_n^{ll'}(a;l_0,p)$ satisfies the equations

$$L_+(n)\,T_n^{ll'}(a;l_0,p)=-2\alpha^l_{n,n+1}\alpha^{l'}_{n,n+1}T^{ll'}_{n+1}(a;l_0,p). \tag{3.8}$$

$$L_-(n)\,T_n^{ll'}(a;l_0,p)=-2\alpha^l_{n-1,n}\alpha^{l'}_{n-1,n}T^{ll'}_{n-1}(a;l_0,p). \tag{3.9}$$

$$K_+(l)\,T_n^{ll'}(a;l_0,p)=-2(2l+1)\,\alpha^{l+1}_{n,n}\gamma_2(l+1;l_0,p)\,aT_n^{l+1,l'}(a;l_0,p). \tag{3.10}$$

$$K_-(l)\,T_n^{ll'}(a;l_0,p)=2(2l+1)\,\alpha^{l}_{n,n}\gamma_2(l;l_0,p)\,aT_n^{l-1,l'}(a;l_0,p). \tag{3.11}$$

We note that all the relations (3.4)–(3.11) can be obtained from the corresponding relations for $L(1,3)$, as given in ref. [4], by a formal application of the limiting procedure (2.7).

We will exploit the equations (3.4)–(3.11) in a way similar to the one used in ref. [4] to find

(a) explicit expressions for $T_l^{ll'}(a; l_0, p)$
(b) explicit expressions for $T_n^{ll_0}(a; l_0, p)$
(c) an iteration procedure by which $T_n^{ll'}(a; l_0, p)$ can be expressed in terms of functions $T_k^{kl'}(a; l_0, p)$.

Consider first item (a). We adopt the procedure of ref. [4], i.e. we make use of the fact that for $l \leqslant l'$ it follows from (3.8) that

$$L_+(l)\, T_l^{ll'}(a; l_0, p) = 0, \tag{3.12}$$

which is a second order differential equation with singularities at $a = 0$ and $a = \infty$. The exponents of the two solutions at $a = 0$ are $\varkappa_1 = l' - l$ and $\varkappa_2 = -l' - l - 1$. Since only solutions bounded at $a = 0$ are admissible, we put

$$T_l^{ll'}(a; l_0, p) = a^{\varkappa_1} \cdot B_l^{ll'}(a; l_0, p), \tag{3.13}$$

where $B_l^{ll'}(a; l_0, p)$ is regular at $a = 0$ and determined by

$$\left[a \frac{d^2}{da^2} + 2(l'+1)\frac{d}{da} + ap^2 + 2il_0 p\right] B_l^{ll'}(a; l_0, p) = 0. \tag{3.14}$$

This equation has the solution [6]

$$B_l^{ll'}(a; l_0, p) = N \cdot e^{ipa}\, \Phi(l' + l_0 + 1, 2l' + 2, -2ipa), \tag{3.15}$$

where $\phi(b, c, x)$ is the confluent hypergeometric series

$$\phi(b, c, x) = 1 + \frac{b}{c}\frac{x}{1} + \frac{b(b+1)}{c(c+1)}\frac{x^2}{2!} + \dots .$$

The normalization constant N is determined by an application of (3.5) and the condition

$$T_{l'}^{l'l'}(0; l_0, p) = 1,$$

which comes from the fact that for $a = 0$ the representation decomposes into a direct sum of representations of the threedimensional rotation group.

The complete expression reads

$$T_l^{ll'}(a; l_0, p) = N \cdot a^{l'-l} \cdot e^{ipa}\, \phi(l' + 1 + l_0, 2l' + 2, -2iap),$$

with
$$N = (-2\,|p|)^{l'-l} \sqrt{\frac{(l'+l_0)!\,(l'-l_0)!\,(2l+1)!\,(l'+l)!}{(l+l_0)!\,(l-l_0)!\,(2l'+1)!\,(l'-l)!\,(2l')!}}. \tag{3.16}$$

Consider next item (b). According to (3.2) the column $T^{ll_0}_{mm'}(\varphi, \theta, \psi, a, \beta, \gamma; l_0, p)$ (l_0, m' fixed) is a sum of functions $T^{ll_0}_{n}(a; l_0, p)$. As we shall see below $T^{ll_0}_{n}(a; l_0, p)$ can be determined explicitly. Consequently the explicit form of all the elements in this column, which form a basis for a linear representation of $E(3)$, is known.

For $|n| \leqslant l_0$ it follows from (3.11) that

$$K_-(l_0)\, T^{l_0 l'}_{n}(a; l_0, p) = 0. \tag{3.17}$$

This equation is in fact identical with

$$L_+(l_0)\, T^{l_0 l}_{l_0}(a; n, p) = 0, \tag{3.18}$$

where $n \geqslant 0$ is assumed. Consequently $T^{l_0 l}_{n}(a; l_0, p) = \text{const.}\, T^{l_0 l}_{l_0}(a; n, p)$ where the r.h.s. is known, according to (3.16). Using the symmetries

$$T^{ll'}_{n}(a; l_0, p) = (-1)^{l'-l}\, T^{l'l}_{n}(a; l_0, p), \tag{3.19}$$

and

$$T^{ll'}_{n}(a; l_0, p) = T^{ll'}_{-n}(a; l_0, -p), \tag{3.20}$$

we thus have

$$T^{ll_0}_{n}(a; l_0, p) = N_1\, e^{ipa}\, a^{l-l_0}\, \phi(l+n+1, 2l+2, -2ipa), \tag{3.21}$$

for all $l \geqslant l_0$ and $|n| \leqslant l_0$.

Consider finally item (c). Because of the length of the explicit calculations we only indicate how the general function $T^{ll'}_{n}(a; l_0, p)$ can be determined from (3.16). We refer to ref. [4] for more details about the procedure. From the recursion relations between associated confluent hypergeometric functions [6] it follows that

$$\left[2na\frac{d}{da} + (l_0+1)\,2ipa + 2n(l'-l+1)\right]\phi(b, c, -2ipa)$$
$$= \chi_1(-2ipa)^2\,\phi(b+1, c+2, -2ipa)$$
$$+ \chi_2(-2ipa)\,\phi(b, c, -2ipa) + \chi_3\,\phi(b-1, c-2, -2ipa),$$

where
$$\chi_1 = \frac{2n(l'+1-l_0)(l'+1+l_0)(l'+l)}{(2l'+2)^2(2l'+1)(2l'+3)},$$

$$\chi_2 = n - l_0 - 1 + \frac{l_0(l-1)}{l'(l'+1)},$$

$$\chi_3 = 2n(l'-l+1),$$

$$b = l'+l_0+1, \quad c = 2l'+2. \tag{3.22}$$

Consequently the functions $T^{ll'}_{l}(a; l_0, p)$ satisfy a relation of the following form:

$$\left[2na\frac{d}{da}+2n+2il_0pa\right]T_l^{ll'}(a;l_0,p)$$
$$=c_1T_{l-1}^{l-1,l'+1}(a;l_0p)+c_0T_{l-1}^{l-1,l'}(a;l_0,p)+c_{-1}T_{l-1}^{l-1,l'-1}(a;l_0,p). \quad (3.23)$$

It is easily seen that by a repeated application of (3.4) and (3.24), the general function $T_n^{ll'}(a;l_0,p)$ can be expressed as a sum of known functions $T_{l-p}^{l-p,l'+\mu}(a;l_0,p)$, where $0\leqslant p\leqslant l-n$, $p\geqslant\mu\geqslant\max(l_0-l',-p)$ (cf. ref. [4]).

Finally we may remark that, in the same way as in the Lorentz case, $T_l^{ll'}(a;l_0,p)$ can be expressed as a linear combination of two solutions of (3.12) corresponding to $\varkappa=\varkappa_2$. One finds

$$T_l^{ll'}(a;l_0,p)=N_2\cdot a^{-l-l'-1}\cdot[e^{ipa}\phi(-l'+l_0,-2l',-2ipa)$$
$$-e^{-ipa}\phi(-l'-l_0,-2l',+2ipa),$$

where the confluent hypergeometric functions are polynomials of degree $l'-l_0$ and $l'+l_0$ respectively.

4. The contraction of the representations of $L(1,3)$

An arbitrary element L of $L(1,3)$ an be expressed as a function of six parameters $(\varphi,\theta,\psi,\alpha,\beta,\gamma)$ similar to (3.1) for $E(3)$:

$$L=R_z(\varphi)\,R_x(\theta)\,R_z(\psi)\,A_z(\alpha)\,R_x(\beta)\,R_z(\gamma), \quad (4.1)$$

where $A_z(\alpha)$ is an acceleration in the z direction with the velocity $v=\operatorname{Tgh}\alpha(c=1)$. The matrix element corresponding to L can, in a basis $|l,m;l_0,\nu\rangle$, be written [4]

$$T_{mm'}^{ll'}(\varphi,\theta,\psi,\alpha,\beta,\gamma;l_0,\nu)=\sum_{n=-\min(l,l')}^{n=\min(l,l')}D_{m,n}^{l}(\varphi,\theta,\psi)\,A_n^{ll'}(\alpha;l_0,\nu)\,D_{n,m'}^{l'}(0,\beta,\gamma). \quad (4.2)$$

In section 3 it was mentioned that the equations for $T_n^{ll'}(a;l_0,p)$ could be constructed from the corresponding ones for $A_n^{ll'}(\alpha;l_0,\nu)$ by taking the limit (2.7). We will now show explicitly that the solutions of these equations are connected by the same limit. This is of course well known for those parts of the solutions which consists of hypergeometric functions and confluent hypergeometric functions respectively. However, it must be shown that the remaining parts of the matrix elements are also connected by the limit (2.7).

To begin with, $T_l^{ll'}(a;l_0,p)$ and $A_l^{ll'}(\alpha;l_0,\nu)$ will be compared. $A_l^{ll'}(\alpha;l_0,\nu)$ can be written as follows:

$$A_l^{ll'}(\alpha;l_0,\nu)=N_3\,(\operatorname{Sh}\alpha)^{l'-l}\cdot e^{\alpha(i\nu-l'-l_0-1)}\cdot F(l'+1+l_0,l'+1-i\nu,2l'+2,1-e^{-2\alpha}),$$

where

$$N_3=(-2)^{l'-l}\sqrt{\frac{(l'+l_0)!\,(l'-l_0)!\,(l'+l)!\,(2l+1)!\,((l+1)^2+\nu^2)\ldots(l'^2+\nu^2)}{(l+l_0)!\,(l-l_0)!\,(2l')!\,(2l'+1)\,(l'-l)!}}. \quad (4.3)$$

$F(a, b, c, x)$ is the hypergeometric series:

$$F(a, b, c, x) = 1 + \frac{a \cdot b}{1 \cdot c}\, x + \frac{a(a+1)\, b(b+1)}{1 \cdot 2 \cdot c(c+1)}\, x^2 + \dots .$$

(Cf. ref. [4]. Equation (4.3) is not explicitly given in [4], but it can readily be derived from expressions given there. In the notation of [4], choose $\sigma = \sigma_1 = l' - 1$, $\varrho = \varrho_1 = \frac{1}{2}(l + l_0 + 1 - i\nu)$ and use the argument $1 - e^{-2\alpha}$ in the hypergeometric function.)

In the limit (2.7) we have for the factors appearing in (4.3):

$$\lim_{(2.7)} (\operatorname{Sh} \lambda\alpha)^{l'-l}\, e^{\lambda\alpha(i\nu - l' - l_0 - 1)} \cdot \sqrt{((l+1)^2 + \nu^2) \dots (l'^2 + \nu^2)} = \alpha^{l'-l}\, |p|^{l'-l}\, e^{ip\alpha}, \qquad (4.4)$$

and

$$\lim_{(2.7)} F(l' + 1 + l_0, l' + 1 - i\nu, 2l' + 2, 1 - e^{-2\lambda\alpha}) = \phi(l' + 1 + l_0, 2l' + 2, -2ip\alpha). \qquad (4.5)$$

From (3.16), (4.3), (4.4) and (4.5) it evidently follows that

$$\lim_{(2.7)} A_l^{ll'}(\lambda\alpha; l_0, \nu) = T_l^{ll'}(\alpha; l_0, p). \qquad (4.6)$$

The contraction is thus valid for these special elements (and thus also for $A_n^{ll_0}(\alpha; l_0, \nu)$ and $T_n^{ll_0}(a; l_0, p)$).

To see that the contraction of an arbitrary element $A_n^{ll'}(\alpha; l_0, \nu)$ gives $T_n^{ll'}(a; l_0, p)$ we note that the following relation between associated hypergeometric functions holds:

$$\left[\frac{1 - e^{-2\alpha}}{2} \cdot \frac{d}{d\alpha} + (i\nu(l_0 + 1) - (l' + l_0 + 1))(1 - e^{-2\alpha}) + 2n(l' - l + 1) \right.$$
$$\left. - 2n(l' - l + \tfrac{1}{2})(1 - e^{-2\alpha})\right] F(r, s, t, 1 - e^{-2\alpha})$$
$$= \gamma_1 (1 - e^{-2\alpha})^2 F(r + 1, s + 1, t + 2, 1 - e^{-2\alpha})$$
$$+ \gamma_2 (1 - e^{-2\alpha}) F(r, s, t, 1 - e^{-2\alpha}) + \gamma_3 F(r - 1, s - 1, t - 2, 1 - e^{-2\alpha}), \qquad (4.7)$$

where $r = l' + l_0 + 1$, $s = l' + 1 - i\nu$, $t = 2l' + 2$. The quantities γ_1, γ_2 and γ_3 are constants which in the limit (2.7) obey

$$\lim_{(2.7)} \lambda^2 \gamma_1 = (-ip)^2 \chi_1, \quad \lim_{(2.7)} \lambda \gamma_2 = (-ip)\, \chi_2, \quad \gamma_3 = \chi_3. \qquad (4.8)$$

If α is replaced by $\lambda\alpha$ and the limit (2.7) is taken, (4.7) is transformed into (3.22), which is of course a consequence of the fact that the confluence, which takes the hypergeometric function into a confluent hypergeometric function, also takes the relations between associated hypergeometric functions into corresponding relations between associated confluent hypergeometric functions. $A_n^{ll'}(\alpha; l_0, \nu)$ is determined from $A_l^{ll'}(\alpha; l_0, \nu)$ by repeated application of the relations

$$\left[2n \operatorname{Sh}\alpha \frac{d}{d\alpha} + 2n \operatorname{Ch}\alpha + 2il_0\nu \operatorname{Sh}\alpha\right] A_n^{ll'}(\alpha; l_0, \nu)$$
$$= \alpha_{n-1,n}^{l}\, \alpha_{n-1,n}^{l'}\, A_{n-1}^{ll'}(\alpha; l_0, \nu) - \alpha_{n,n+1}^{l}\, \alpha_{n,n+1}^{l'}\, A_{n+1}^{ll'}(\alpha; l_0, \nu),$$

(corresponding to (3.4)) and

$$\left[2n \operatorname{Sh}\alpha \frac{d}{d\alpha} + 2n \operatorname{Ch}\alpha + 2il_0\nu \operatorname{Sh}\alpha\right] A_n^{ll'}(\alpha; l_0, \nu)$$
$$= d_1 A_{l-1}^{l-1,l'+1}(\alpha; l_0, \nu) + d_0 A_{l-1}^{l-1,l'}(\alpha; l_0, \nu) + d_{-1} A_{l-1}^{l-1,l'-1}(\alpha; l_0, \nu),$$

(corresponding to (3.23)). Since $A_n^{ll'}(\alpha; l_0, \nu)$ and $T_n^{ll'}(a; l_0, p)$ can be obtained from $A_l^{ll'}(\alpha; l_0, \nu)$ and $T_l^{ll'}(a; l_0, p)$ respectively, by applying, a finite number of times, relations which are connected by the limit (2.7) it follows that

$$\lim_{(2.7)} A_n^{ll'}(\lambda\alpha; l_0, \nu) = T_n^{ll'}(\alpha; l_0, p), \tag{4.9}$$

i.e. the matrix elements in a basis $|l, m; l_0, p\rangle$ of a general element E of $E(3)$ in a unitary representation (l_0, p) can be obtained from the corresponding elements in a unitary representation (l_0, ν) of $L(1, 3)$, by an application of the limiting procedure (2.7).

If we go down one step in the space dimensions and consider the contraction of the threedimensional Lorentz group $L(1, 2)$ to the twodimensional Euclidean group $E(2)$, we find that in this case the matrix elements of an irreducible unitary representation of $L(1, 2)$ belonging to the continuous class [7] can be contracted, by a procedure analogous to (2.7), to give the matrix elements in a unitary irreducible representation of $E(2)$. This follows from an application of Hansens formula for the Bessel function as a limit of hypergeometric functions [8].

5. A choice of representation space suggested by the contraction

According to § 11 of ref. [2] an explicit realization of the basis functions $|l, m; l_0, \nu\rangle$ is

$$|l, m; l_0, \nu\rangle = N(l, l_0, \nu)\, D_{l_0, m}^{l}(\varphi, \theta, \psi). \tag{5.1}$$

We refer to [2] for the general formula for the operator representing an arbitrary Lorentz transformation in this realization. However, if V_α, V_β and V_γ are the operators corresponding to accelerations in the z, x and y directions with velocities $\operatorname{Tgh}\alpha$, $\operatorname{Tgh}\beta$ and $\operatorname{Tgh}\gamma$ respectively, the explicit expressions for the action of V_α and V_β in this basis is given by

$$V_\alpha D_{l_0, m}^{l}(\varphi, \theta, \psi) = (\operatorname{Ch}\alpha - \cos\theta \operatorname{Sh}\alpha)^{i\nu - 1} D_{l_0, m}^{l}(\varphi, \theta', \psi),$$

where
$$\cos\theta' = \frac{\cos\theta \operatorname{Ch}\alpha - \operatorname{Sh}\alpha}{\operatorname{Ch}\alpha - \cos\theta \operatorname{Sh}\alpha}, \tag{5.2}$$

and $$V_\beta D^l_{l_0,m}(\varphi,\theta,\psi)=(\operatorname{Ch}\beta-\sin\theta\sin\psi\operatorname{Sh}\beta)^{i\nu-1}D^l_{l_0,m}(\varphi'',\theta'',\psi''),$$

where $$\cos\theta''=\frac{\cos\theta}{\operatorname{Ch}\beta-\sin\theta\sin\psi\operatorname{Sh}\beta},$$

$$\varphi''=\varphi+\operatorname{arctg}\left\{\frac{\cos\theta\cos\psi\operatorname{Sh}\beta}{\sin\theta\operatorname{Ch}\beta-\sin\psi\operatorname{Sh}\beta}\right\},$$

$$\psi''=\psi+\operatorname{arctg}\left\{\frac{\sin\theta\sin2\psi\operatorname{Sh}^2\frac{\beta}{2}-\operatorname{Sh}\beta\cos\psi}{\sin\theta\operatorname{Ch}^2\frac{\beta}{2}-\sin\theta\cos2\psi\operatorname{Sh}^2\frac{\beta}{2}-\operatorname{Sh}\beta\sin\psi}\right\}. \tag{5.3}$$

There is a completely analogous formula for V_γ:

$$V_\gamma D^l_{l_0,m}(\varphi,\theta,\psi)=(\operatorname{Ch}\gamma-\sin\theta\cos\psi\operatorname{Sh}\gamma)^{i\nu-1}D^l_{l_0,m}(\varphi''',\theta''',\psi'''). \tag{5.4}$$

In this realization every element L of $L(1,3)$ is associated with a transformation of a point (φ,θ,ψ) in "carrier space". The operator corresponding to a rotation changes (φ,θ,ψ) and the new arguments are determined by multiplication from the right in the D-functions (cf. [2] and equation (5.9) below).

We have

$$\lim_{\lambda\to0}\theta'(\lambda\alpha)=\theta,$$

$$\lim_{\lambda\to0}\theta''(\lambda\beta)=\theta,\quad \lim_{\lambda\to0}\varphi''(\lambda\beta)=\varphi,\quad \lim_{\lambda\to0}\psi''(\lambda\beta)=\psi.$$

$$\lim_{\lambda\to0}\theta'''(\lambda\gamma)=\theta,\quad \lim_{\lambda\to0}\varphi'''(\lambda\gamma)=\varphi,\quad \lim_{\lambda\to0}\psi'''(\lambda\gamma)=\psi. \tag{5.5}$$

A formal application of the limit (2.7) to (5.2), (5.3) and (5.4) then gives

$$\lim_{(2.7)}V_{\lambda\alpha}D^l_{l_0,m}(\varphi,\theta,\psi)=e^{-ip\alpha\cos\theta}D^l_{l_0,m}(\varphi,\theta,\psi),$$

$$\lim_{(2.7)}V_{\lambda\beta}D^l_{l_0,m}(\varphi,\theta,\psi)=e^{-ip\alpha\sin\theta\sin\psi}D^l_{l_0,m}(\varphi,\theta,\psi),$$

$$\lim_{(2.7)}V_{\lambda\gamma}D^l_{l_0,m}(\varphi,\theta,\psi)=e^{-ip\alpha\sin\theta\cos\psi}D^l_{l_0,m}(\varphi,\theta,\psi). \tag{5.6}$$

As is implied by (5.6) we can now define a representation (l_0,p) of $E(3)$ as a "multiplier representation".

Every element E of $E(3)$ can be written as (5.7)

$$E=T\cdot R$$

where T is a translation and R is a rotation. The operators representing T and R will be denoted by O_T and O_R respectively. As representation space we choose the set $\{D^l_{l_0,m}(\varphi,\theta,\psi)\}$ where $l=l_0, l_0+1 \dots$ and $m=-l,\dots l$. The action of O_T and O_R is defined by

$$O_T D(R_0)=e^{-ip(T,R_0)}D(R_0), \tag{5.8}$$

$$O_R D(R_0)=D(R_0 R), \tag{5.9}$$

where $(T,R_0)=T_x\sin\theta_0\sin\psi_0+T_y\sin\theta_0\cos\psi_0+T_z\cos\theta_0$ (R_0 is the rotation described by $(\varphi_0,\theta_0,\psi_0)$). With every rotation is associated a transformation of the point $(\varphi_0,\theta_0,\psi_0)$ in carrier space according to (5.9), while the translations give a multiplier according to (5.8). The operator $O_E\equiv O_T\cdot O_R$ gives

$$O_E D(R_0)=O_T O_R D(R_0)=O_T D(R_0R)=e^{-ip(T,R_0)}D(R_0R). \tag{5.10}$$

The product of two operators corresponding to arbitrary elements E_1 and E_2 of $E(3)$ gives

$$O_{E_1}O_{E_2}D(R_0)=e^{-ip[(T_1,R_0)+(T_2,R_0R_1)]}D(R_0R_1R_2). \tag{5.11}$$

If we put $$E_1\cdot E_2=E=T\cdot R,$$

it follows that $$T=T_1R_1T_2R_1^{-1},\quad R=R_1\cdot R_2,$$

and $$O_E D(R_0)=e^{-ip[(T_1,R_0)+(R_1T_2R_1^{-1},R_0)]}D(R_0R_1R_2). \tag{5.12}$$

It is easy to show that

$$(T_2,R_0R_1)=(R_1T_2R_1^{-1},R_0).$$

From (5.11) and (5.12) it then follows that

$$O_{E_1}\cdot O_{E_2}=O_{E_1\cdot E_2}$$

and thus the operators O_E actually give a representation (l_0,p) of $E(3)$.

From this realization the following integral representation of $T^{ll'}_n(a;l_0,p)$ is obtained:

$$T^{ll'}_n(a;l_0,p)=N(l,l',n;l_0,p)\int_0^\pi D^l_{l_0,n}(0,\theta,0)\,e^{-ipa\cos\theta}D^{l'}_{l_0,n}(0,\theta,0)\sin\theta\,d\theta. \tag{5.13}$$

For $n=l$ equation (5.13) can be recognized as being essentially Eulers integral formula for the confluent hypergeometric function, [6].

The realization given above, or very similar ones, arises naturally in many different approaches to the representation theory of $E(3)$. The main purpose of this section is to point out the connection of such realizations with the multiplier representations of $L(1,3)$.

ACKNOWLEDGEMENT

The author wishes to thank professor Nils Svartholm for his kind interest and valuable suggestions.

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APPENDIX

Differential operators for $p_{\pm}$, p_3, $\mathbf{p}^2$ and $\mathbf{p}\cdot\mathbf{M}$

$$p_+ = \frac{i\,e^{-i\varphi}}{a}\left\{\frac{1}{s_2}\frac{\partial}{\partial\varphi} + ic_2\frac{\partial}{\partial\theta} - \left[\frac{c_4}{s_4}(c_3 - is_3c_2) + \frac{c_2}{s_2}\right]\frac{\partial}{\partial\psi} + is_2a\frac{\partial}{\partial a} - (s_3 + ic_3c_2)\frac{\partial}{\partial\beta}\right.$$
$$\left. + \frac{1}{s_4}(c_3 - is_3c_2)\frac{\partial}{\partial\gamma}\right\},$$

$$p_- = \frac{i\,e^{i\varphi}}{a}\left\{\frac{1}{s_2}\frac{\partial}{\partial\varphi} - ic_2\frac{\partial}{\partial\theta} - \left[\frac{c_4}{s_4}(c_3 + is_3c_2) + \frac{c_2}{s_2}\right]\frac{\partial}{\partial\psi} - is_2a\frac{\partial}{\partial a} - (s_3 - ic_3c_2)\frac{\partial}{\partial\beta}\right.$$
$$\left. + \frac{1}{s_4}(c_3 + is_3c_2)\frac{\partial}{\partial\gamma}\right\},$$

$$p_3 = \frac{i}{a}\left\{-s_2\frac{\partial}{\partial\theta} - \frac{s_2s_3c_4}{s_4}\frac{\partial}{\partial\psi} + c_2a\frac{\partial}{\partial a} + s_2c_3\frac{\partial}{\partial\beta} + \frac{s_2s_3}{s_4}\frac{\partial}{\partial\gamma}\right\},$$

$$\mathbf{p}^2 = \frac{1}{a^2}\left\{-\frac{1}{s_2^2}\frac{\partial^2}{\partial\varphi^2} - \frac{\partial^2}{\partial\theta^2} - \left[\left(\frac{c_2}{s_2}\right)^2 + \left(\frac{c_4}{s_4}\right)^2 + 2\frac{c_2c_3c_4}{s_2s_4}\right]\frac{\partial^2}{\partial\psi^2} - a^2\frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial\beta^2} - \frac{1}{s_4^2}\frac{\partial^2}{\partial\gamma^2}\right.$$
$$+ 2\left(\frac{c_3c_4}{s_2s_4} + \frac{c_2}{s_2^2}\right)\frac{\partial^2}{\partial\varphi\,\partial\psi} + 2\frac{s_3}{s_2}\frac{\partial^2}{\partial\varphi\,\partial\beta} - 2\frac{c_3}{s_2s_4}\frac{\partial^2}{\partial\varphi\,\partial\gamma} - 2\frac{s_3c_4}{s_4}\frac{\partial^2}{\partial\theta\,\partial\psi} + 2c_3\frac{\partial^2}{\partial\theta\,\partial\beta}$$
$$\left. + 2\frac{s_3}{s_4}\frac{\partial^2}{\partial\theta\,\partial\gamma} - 2\frac{s_2s_3}{c_2}\frac{\partial^2}{\partial\beta\,\partial\psi} + 2\left(\frac{c_2c_3}{s_2s_4} + \frac{c_4}{s_4^2}\right)\frac{\partial^2}{\partial\psi\,\partial\gamma} - \frac{c_2}{s_2}\frac{\partial}{\partial\theta} - 2a\frac{\partial}{\partial a} - \frac{c_4}{s_4}\frac{\partial}{\partial\beta}\right\},$$

$$\mathbf{p}\cdot\mathbf{M} = \frac{1}{a}\left\{-\frac{c_2c_4s_3}{s_2s_4}\frac{\partial^2}{\partial\psi^2} + \frac{s_3c_4}{s_2s_4}\frac{\partial^2}{\partial\varphi\,\partial\psi} - \frac{c_3}{s_2}\frac{\partial^2}{\partial\varphi\,\partial\beta} - \frac{s_3}{s_2s_4}\frac{\partial^2}{\partial\varphi\,\partial\gamma} + \frac{c_3c_4}{s_4}\frac{\partial^2}{\partial\theta\,\partial\psi} + s_3\frac{\partial^2}{\partial\theta\,\partial\beta}\right.$$
$$\left. - \frac{c_3}{s_4}\frac{\partial^2}{\partial\theta\,\partial\gamma} - a\frac{\partial^2}{\partial\psi\,\partial a} + \frac{c_2c_3}{s_2}\frac{\partial^2}{\partial\beta\,\partial\psi} + \frac{c_2s_3}{s_2s_4}\frac{\partial^2}{\partial\psi\,\partial\gamma} - \frac{\partial}{\partial\psi}\right\},$$

where $$s_1 = \sin\varphi,\quad s_2 = \sin\theta,\quad s_3 = \sin\psi,\quad s_4 = \sin\beta,$$

$$c_1 = \cos\varphi,\quad c_2 = \cos\theta,\quad c_3 = \cos\psi,\quad c_4 = \cos\beta.$$

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Tryckt den 29 oktober 1965

Uppsala 1965. Almqvist & Wiksells Boktryckeri AB

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GÖTEBORG · UPPSALA

1965

ARKIV FÖR FYSIK Band 30 nr 31

3.65 10 5 Communicated 2 June 1965 by L. HULTHÉN and N. SVARTHOLM

Construction of representations of the inhomogeneous Lorentz group by means of contraction of representations of the (1+4) de Sitter group

By STAFFAN STRÖM

ABSTRACT

If the Lie algebra of the $(1+4)$ de Sitter group is contracted, in the sense of Inönü and Wigner, with respect to the Lie algebra of the homogeneous Lorentz group, the resulting algebra is the Lie algebra of the inhomogeneous Lorentz group. In the present work we study the corresponding process of contraction in the representations of the Lie algebra of the de Sitter group. It is shown that the infinitesimal generators of all the unitary irreducible representations of the inhomogeneous Lorentz group can be constructed in this way, in terms of an angular momentum basis.

1. Introduction

One characteristic feature of the contraction [1] of a Lie algebra with respect to any of its subalgebras is that the algebra thus obtained has an invariant abelian subalgebra. The contraction therefore results in a very marked change of structure of the Lie algebra. In terms of the corresponding Lie groups this means e.g. that by contraction of a simple or semisimple Lie group we arrive at a non-semisimple Lie group.

From a physical point of view one is primarily interested in the relation between the unitary irreducible representations of the two groups which are related to each other through a contraction. One way of obtaining representations of the group obtained by contraction, from representations of the contracted group, is to consider, together with the process of contraction, the limit of special sets of representations of the contracted group [1]. Such constructions may provide simple ways of obtaining representations of non-semisimple groups from representations of simple ones.

In the present work we will study a case where the complete classification of the unitary irreducible representations of the two groups involved, the $(1+4)$ de Sitter group $L(1,4)$ and the (proper, orthochronous) inhomogeneous Lorentz group $P(1,3)$, is known. The emphasis will be on the actual construction of the representations of $P(1,3)$. (We will not discuss possible physical implications. For a discussion of such questions, see references [2] and [3].) The complete classification of the unitary irreducible representations of $L(1,4)$ has been given by Dixmier [4]. The relevant results from that work are listed in section II. In section III we perform a change of basis in the representation space given in ref. [4], which is necessary for the later developments. In section IV the particular constructions by means of which one gets all the different unitary representations of $P(1,3)$ (in infinitesimal form) are given.

II. The $L(1,4)$ and its unitary representations

The Lie algebra of $L(1,4)$ has ten basis elements which may be chosen as $\mathbf{M}=(M_1, M_2, M_3)$, $\mathbf{N}=(N_1, N_2, N_3)$, $\mathbf{P}=(P_1, P_2, P_2)$ and P_0 which satisfy the commutation relations

$$\begin{aligned}
&[M_k, M_l]=iM_m,\ [N_k, N_l]=-iM_m,\ [P_k, P_l]=iM_m,\\
&[M_k, N_l]=[N_k, M_l]=iN_m,\\
&[M_k, P_l]=[P_k, M_l]=iP_m,\ k,l,m \text{ in cyclic perm.}\\
&[M_k, N_k]=[M_k, P_k]=[M_k, P_0]=0,\\
&[P_0, N_k]=iP_k,\ [P_0, P_k]=iN_k,\\
&[P_k, N_l]=i\delta_{kl}P_0.
\end{aligned}\tag{2.1}$$

$\mathbf{M}, \mathbf{N}, \mathbf{P}$ and P_0 are assumed to be hermitian. $\mathbf{M}$ and $\mathbf{N}$ form a subalgebra, the Lie algebra of the homogeneous Lorentz group $L(1,3)$. $\mathbf{M}$ and $\mathbf{P}$ form another subalgebra which is the Lie algebra of the fourdimensional rotation group $R(4)$. We will use the same notation $\mathbf{M}, \mathbf{N}\ldots$ etc. for the corresponding operators acting in the representation space. For all the unitary representations of $L(1,4)$ the representation space $\mathcal{H}$ can be taken as the direct sum of representation spaces $\mathcal{H}_{k,k'}$ for a $(2k+1)(2k'+1)$-dimensional representation of $R(4)$:

$$\mathcal{H}=\sum_{k,k'}\oplus\mathcal{H}_{k,k'}.\tag{2.2}$$

The values of k and k' which are to be summed over will be specified later. In $\mathcal{H}_{k,k'}$ one can choose a basis $|k,\mu,k'\mu'\rangle$ where $(\mathbf{M}+\mathbf{P})^2$, $(\mathbf{M}-\mathbf{P})^2$, $(\mathbf{M}+\mathbf{P})_3$ and $(\mathbf{M}-\mathbf{P})_3$ are diagonal. Reviewing and rewriting the results of ref. [4] we have in this basis:

$$\begin{aligned}
&\tfrac{1}{2}(M_3-P_3)|k,\mu,k',\mu'\rangle=\mu|k,\mu,k',\mu'\rangle,\\
&\tfrac{1}{2}(M_+-P_+)|k,\mu,k',\mu'\rangle=\sqrt{(k-\mu)(k+\mu+1)}\,|k,\mu+1,k',\mu'\rangle,\\
&\tfrac{1}{2}(M_--P_-)|k,\mu,k',\mu'\rangle=\sqrt{(k+\mu)(k-\mu+1)}\,|k,\mu-1,k',\mu'\rangle,\\
&\tfrac{1}{2}(M_3+P_3)|k,\mu,k',\mu'\rangle=\mu'|k,\mu,k',\mu'\rangle,\\
&\tfrac{1}{2}(M_++P_+)|k,\mu,k',\mu'\rangle=\sqrt{(k'-\mu')(k'+\mu'+1)}\,|k,\mu,k',\mu'+1\rangle,\\
&\tfrac{1}{2}(M_-+P_-)|k,\mu,k',\mu'\rangle=\sqrt{(k'+\mu')(k'-\mu'+1)}\,|k,\mu,k',\mu'-1\rangle,
\end{aligned}$$

$$N_+|k,\mu,k',\mu'\rangle = i\{\sqrt{(k+\mu+1)(k'+\mu'+1)}\,A_{k,k'}|k+\tfrac{1}{2},\mu+\tfrac{1}{2},k'+\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle$$

$$+\sqrt{(k-\mu)(k'+\mu'+1)}\,B_{k,k'}|k-\tfrac{1}{2},\mu+\tfrac{1}{2},k'+\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle$$

$$+\sqrt{(k+\mu+1)(k'-\mu')}\,C_{k,k'}|k+\tfrac{1}{2},\mu+\tfrac{1}{2},k'-\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle$$

$$+\sqrt{(k-\mu)(k'-\mu')}\,D_{k,k'}|k-\tfrac{1}{2},\mu+\tfrac{1}{2},k'-\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle\},$$

$$N_-|k,\mu,k',\mu'\rangle = i\{-\sqrt{(k-\mu+1)(k'-\mu'+1)}\,A_{k,k'}|k+\tfrac{1}{2},\mu-\tfrac{1}{2},k'+\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$+\sqrt{(k+\mu)(k'-\mu'+1)}\,B_{k,k'}|k-\tfrac{1}{2},\mu-\tfrac{1}{2},k'+\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$+\sqrt{(k-\mu+1)(k'+\mu')}\,C_{k,k'}|k+\tfrac{1}{2},\mu-\tfrac{1}{2},k'-\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$-\sqrt{(k+\mu)(k'+\mu')}\,D_{k,k'}|k-\tfrac{1}{2},\mu-\tfrac{1}{2},k'-\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle\},$$

$$N_3|k,\mu,k',\mu'\rangle$$

$$=\frac{i}{2}\{-A_{k,k'}[\sqrt{(k+\mu+1)(k'-\mu'+1)}|k+\tfrac{1}{2},\mu+\tfrac{1}{2},k'+\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$+\sqrt{k-\mu+1)(k'+\mu'+1)}|k+\tfrac{1}{2},\mu-\tfrac{1}{2},k'+\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle]$$

$$+B_{k,k'}[-\sqrt{(k-\mu)(k'-\mu'+1)}|k-\tfrac{1}{2},\mu+\tfrac{1}{2},k'+\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$+\sqrt{(k+\mu)(k'+\mu'+1)}|k-\tfrac{1}{2},\mu-\tfrac{1}{2},k'+\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle]$$

$$+C_{k,k'}[\sqrt{(k+\mu+1)(k'+\mu')}|k+\tfrac{1}{2},\mu+\tfrac{1}{2},k'-\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$-\sqrt{(k-\mu+1)(k'-\mu')}|k+\tfrac{1}{2},\mu-\tfrac{1}{2},k'-\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle]$$

$$+D_{k,k'}[\sqrt{(k-\mu)(k'+\mu')}|k-\tfrac{1}{2},\mu+\tfrac{1}{2},k'-\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$+\sqrt{(k+\mu)(k'-\mu')}|k-\tfrac{1}{2},\mu-\tfrac{1}{2},k'-\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle]\},$$

$$P_0|k,\mu,k',\mu'\rangle$$

$$=\frac{i}{2}\{A_{k,k'}[\sqrt{(k+\mu+1)(k'-\mu'+1)}|k+\tfrac{1}{2},\mu+\tfrac{1}{2},k'+\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$-\sqrt{(k-\mu+1)(k'+\nu'+1)}|k+\tfrac{1}{2},\mu-\tfrac{1}{2},k'+\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle]$$

$$+B_{k,k'}[\sqrt{(k-\mu)(k'-\mu'+1)}|k-\tfrac{1}{2},\mu+\tfrac{1}{2},k'+\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$+\sqrt{(k+\mu)(k'+\mu'+1)}|k-\tfrac{1}{2},\mu-\tfrac{1}{2},k'+\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle]$$

$$-C_{k,k'}[\sqrt{(k+\mu+1)(k'+\mu')}|k+\tfrac{1}{2},\mu+\tfrac{1}{2},k'-\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$+\sqrt{(k-\mu+1)(k'-\mu')}|k+\tfrac{1}{2},\mu-\tfrac{1}{2},k'-\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle]$$

$$+D_{k,k'}[-\sqrt{(k-\mu)(k'+\mu')}|k-\tfrac{1}{2},\mu+\tfrac{1}{2},k'-\tfrac{1}{2},\mu'-\tfrac{1}{2}\rangle$$

$$+\sqrt{(k+\mu)(k'-\mu')}|k-\tfrac{1}{2},\mu-\tfrac{1}{2},k'-\tfrac{1}{2},\mu'+\tfrac{1}{2}\rangle]\},$$

where $M_\pm = M_1 \pm iM_2,\ P_\pm = P_1 \pm iP_2,\ N_\pm = N_1 \pm iN_2.$ (2.3)

The constants $A_{k,k'} \dots D_{k,k'}$ are determined by the commutation relations, the unitarity and the irreducibility conditions. They can be chosen as real and we have

$$D_{k,k'} = A_{k-\frac{1}{2},\, k'-\frac{1}{2}}, \quad C_{k,k} = -B_{k+\frac{1}{2},\, k'-\frac{1}{2}}. \tag{2.4}$$

They have different values in the different classes of unitary representations (see below). In the following the parameters $1 = k + k' + 1$ and $n = k' - k$ will be used frequently and we write

$$A_{k,k'} = a_{l,n}, \quad B_{k,k'} = b_{l,n}, \quad C_{k,k'} = c_{l,n}, \quad D_{k,k'} = d_{l,n},$$

and thus $\quad d_{l,n} = a_{l-1,n}, \quad c_{l,n} = -b_{l,n-1}.$

$L(1,4)$ has two independent invariants which can be chosen as

$$\left.\begin{aligned} \Omega &= P_0^2 - \mathbf{P}^2 - (\mathbf{M}^2 - \mathbf{N}^2), \\ \Omega' &= (\mathbf{M}\cdot\mathbf{P})^2 - (P_0\mathbf{M} - \mathbf{P}\times\mathbf{N})^2 - (\mathbf{M}\cdot\mathbf{N})^2. \end{aligned}\right\} \tag{2.5}$$

The unitary irreducible representations of $L(1,4)$ may be separated into two main classes we will call the continuous and the discrete classes and within each class three subclasses will be specified:

1. *The continuous class*

Among the parameters which specify the representations of this class there is one, σ, which can take all values in a semi-infinite interval. The three subclasses are:

(*a*) the representations $\nu_{r,\sigma}$ with $r = 1, 2, 3 \dots$ and $\sigma > 0$, where $r = \min(k + k')$,

(*b*) the representations $\nu_{r,\sigma}$ with $r = \frac{1}{2}, \frac{3}{2}, \dots$ and $\sigma > \frac{1}{4}$ where $r = \min(k + k')$,

(*c*) the representations $\nu_{0,\sigma}$ with $\sigma > -2$. Here $k = k' \geqslant 0$.

The domain ν of Fig. 1. gives the values of (k, k') which occur in the sum (2.2). In terms of the parameters $l = k + k' + 1$ and $n = k' - k$ the values of $a_{l,n}$ and $b_{l,n}$ are given by

$$\left.\begin{aligned} a_{l,n} &= \sqrt{\frac{(l-r)(l+r+1)(l(l+1)+\sigma)}{(l-n)(l-n+1)(l+n)(l+n+1)}}, \\ b_{l,n} &= \sqrt{\frac{(r-n)(r+n+1)(n(n+1)+\sigma)}{(l-n-1)(l-n)(l+n)(l+n+1)}}. \end{aligned}\right\} \tag{2.6}$$

To get the values for case (*c*) we shall put $r = 0$, $n = 0$. Then $b_{l,0} = 0$. The values of the invariants are

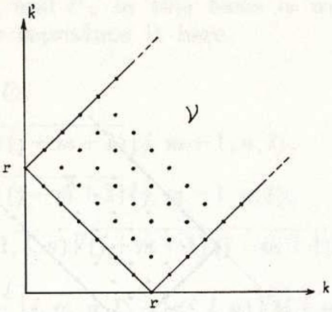

Fig. 1

$$\left.\begin{aligned} \nu_{r,\sigma}(\Omega) &= -r(r+1)+2+\sigma) \\ \nu_{r,\sigma}(\Omega') &= -r(r+1)\,\sigma. \end{aligned}\right\} \tag{2.7}$$

2. *The discrete class*

The representations of this class are characterized by parameters which take discrete values. The three subclasses are

(*a*) the representations $\pi^+_{r,q}$ with $r=\frac{1}{2}, 1, \frac{3}{2}, 2\ldots$
$q=r, r-1, \ldots 1$ or $\frac{1}{2}$ where $r \geqslant n \geqslant q$, $r=\min(k+k')$,

(*b*) the representations $\pi^-_{r,q}$ with $r=\frac{1}{2}, 1, \frac{3}{2}, 2\ldots$
$q=r, r-1, \ldots 1$ or $\frac{1}{2}$ where $r \geqslant -n \geqslant q$, $r=\min(k+k')$,

(*c*) the representations $\pi_{r,o}$ with $r=1, 2, 3, \ldots$.
Here $2k=2k' \geqslant r$.

The domains π of Figs. 2, 3 and 4 give the values of (k, k') which occur in the sum (2,2) in the cases (*a*), (*b*) and (*c*) respectively. The values of $a_{l,n}$ and $b_{l,n}$ are given by

$$\left.\begin{aligned} a_{l,n} &= \sqrt{\frac{(l-r)\,(l+r+1)\,(l+q)\,(l-q+1)}{(l-n)\,(l-n+1)\,(l+n)\,(l+n+1)}}, \\ b_{l,n} &= \sqrt{\frac{(r-n)\,(r+n+1)\,(n+q)\,(n-q+1)}{(l-n-1)\,(l-n)\,(l+n)\,(l+n+1)}}. \end{aligned}\right\} \tag{2.8}$$

To get the values for case (*c*) we shall put $q=1$, $n=0$. Then $b_{l,o}=0$. The values of the invariants are

$$\left.\begin{aligned} \pi_{r,q}(\Omega) &= -r(r+1)-q(q-1)+2, \\ \pi_{r,q}(\Omega') &= p(p+1)\,q(q-1), \end{aligned}\right\} \tag{2.9}$$

where $\pi_{r,q}$ stands for $\pi^{\pm}_{r,q}$ or $\pi_{r,o}$. In the latter case we shall put $q=1$ in (2.9).

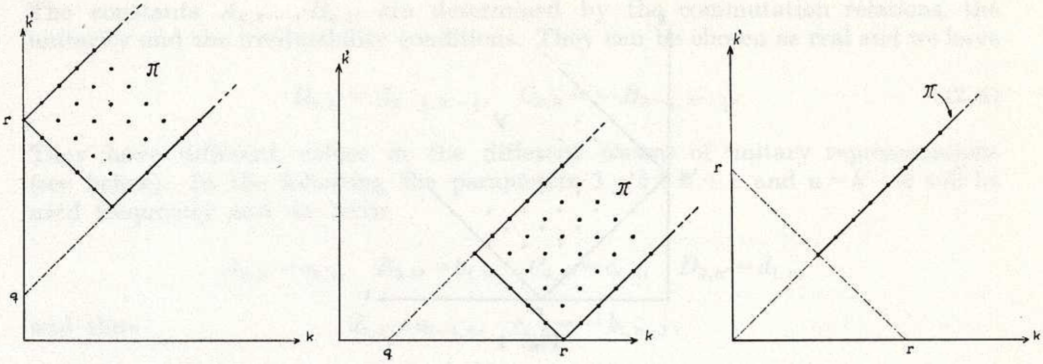

Fig. 2 Fig. 3 Fig. 4

We will now consider the contraction of $L(1,4)$ with respect to $L(1,3)$. Put

$$\mathbf{p}=\lambda\mathbf{P},\quad p_o=\lambda P_o.$$

Then
$$[p_o,\mathbf{p}]=i\lambda^2\mathbf{N},\quad [p_k,p_l]=i\lambda^2 M_m. \tag{2.10}$$

In the remaining commutation relations in (2,1), $\mathbf{P}$ and P_o are simply replaced by $\mathbf{p}$ and p_o. In the limit of $\lambda\to 0$, we have from (2,10):

$$[p_o,\mathbf{p}]=0,\quad [p_k,p_l]=0, \tag{2.11}$$

i.e. in the limit of $\lambda\to 0, \mathbf{M},\mathbf{N},\mathbf{p}$ and p_o form the Lie algebra of $P(1,3)$. Put

$$\left.\begin{aligned}\omega\ &=p_o^2-\mathbf{p}^2,\\ \omega'&=(\mathbf{M}\cdot\mathbf{p})^2-(p_o\mathbf{M}-\mathbf{p}\times\mathbf{N})^2.\end{aligned}\right\} \tag{2.12}$$

ω and ω' are invariants of $P(1,3)$ (for some representations there are further sign invariants). A comparison of (2,5) and (2,12) then shows that formally we have

$$\lim_{\lambda\to 0}\lambda^2\Omega=\omega,\quad \lim_{\lambda\to 0}\lambda^2\Omega'=\omega'. \tag{2.13}$$

III. A change of basis in the representation space

Before we study the process of contraction in the representations we want to change to a basis where $\mathbf{M}^2$ and M_3 are diagonal. Since the contraction takes $R(4)$ into $E(3)$ we expect to get the representation space of $P(1,3)$ as a sum of integrals of representation spaces of $E(3)$. In the latter we want to use a basis where $\mathbf{M}^2$, M_3, $\mathbf{p}^2$ and $\mathbf{p}\cdot\mathbf{M}$ are diagonal. The basis vectors in a representation space of $L(1,4)$ where $\mathbf{M}^2$, M_3, $(\mathbf{M}+\mathbf{P})^2$ and $(\mathbf{M}-\mathbf{P})^2$ are diagonal will be denoted by $|j,m,n,l\rangle$ (instead of the more common notation $|j,m,k,k'\rangle$).

The action of $M_3, M_\pm, P_3$ and $P_\pm$ in this basis is well known [5]. For the convenience of the reader we reproduce it here.

$$
\begin{aligned}
M_3\,|j,m,n,l\rangle &= m\,|j,m,l\rangle,\\
M_+\,|j,m,n,l\rangle &= \sqrt{(j-m)\,(j+m+1)}\,|j,m+1,n,l\rangle,\\
M_-\,|j,m,n,l\rangle &= \sqrt{(j+m)\,(j-m+1)}\,|j,m-1,n,l\rangle,\\
P_3\;\,|j,m,n,l\rangle &= -\alpha(j+1,l,n)\sqrt{(j+m+1)\,(j-m+1)}\,|j+1,m,n,l\rangle\\
&\quad+\frac{m\cdot n\cdot l}{j(j+1)}\,|j,m,n,l\rangle-\alpha(j,l,n)\sqrt{(j+m)\,(j-m)}\,|j-1,m,n,l\rangle,\\
P_+\;\,|j,m,n,l\rangle &= \alpha(j+1,l,\mathrm{n})\sqrt{(j+m+1)\,(j+m+2)}\,|j+1,m+1,n,l\rangle\\
&\quad+\frac{l\cdot n}{j(j+1)}\sqrt{(j-m)\,(j+m+1)}\,|j,m+1,n,l\rangle\\
&\quad-\alpha(j,l,n)\sqrt{(j-m)\,(j-m-1)}\,|j-1,m+1,n,l\rangle,\\
P_-\;\,|j,m,n,l\rangle &= -\alpha(j+1,l,n)\sqrt{(j-m+2)\,(j-m+1)}\,|j+1,m-1,n,l\rangle\\
&\quad+\frac{l\cdot n}{j(+1)}\sqrt{(j+m)\,(j-m+1)}\,|j,m-1,n,l\rangle\\
&\quad+\alpha(j,l,n)\sqrt{(j+m)\,(j+m-1)}\,|j-1,m-1,n,l\rangle,
\end{aligned}
$$

where
$$\alpha(j,l,n)=\frac{1}{j}\sqrt{\frac{(j^2-n^2)\,(l^2-(j+1)^2)}{(2j+1)\,(2j-1)}}. \tag{3.1}$$

Because of the commutation relations

$$[P_o,P_3]=iN_3,\quad [N_3,M_\pm]=\pm N_\pm. \tag{3.2}$$

$N_3\,|j,m,n,l\rangle$ and $N_\pm\,|j,m,n,l\rangle$ can be determined if the expression for $P_o\,|j,m,n,l\rangle$ is known. To determine $P_o\,|j,m,n,l\rangle$ we note that from the commutation relations

$$[P_o,M_3]=0,\quad [P_o,M_\pm]=0,$$

and the expression for $P_o\,|k,\mu,k',\mu'\rangle$ in (2.3) it follows that

$$
\begin{aligned}
P_o\,|j,m,l,n\rangle = a_{l,n}\cdot a(j,l,n)\,|j,m,n,l+1\rangle&\\
+b_{l,n}\cdot b(j,l,n)\,|j,m,n+1,l\rangle&\\
+c_{l,n}\cdot c(j,l,n)\,|j,m,n-1,l\rangle&\\
+d_{l,n}\cdot d(j,l,n)\,|j,m,n,l-1\rangle.&
\end{aligned} \tag{3.3}
$$

We note that a, b, c and d are independent of m. They can be determined in the following way. The bases $|j,m,n,l\rangle$ and $|k,\mu,k',\mu'\rangle$ are connected by the Clebsch–Gordon expansion:

$$|j, m, n, l\rangle = \sum_{\substack{\mu \\ \mu+\mu'=m}} \langle k, \mu, k', \mu' \,|\, j, m\rangle \,|\, k, \mu, k', \mu'\rangle. \tag{3.4}$$

The inverse relation is

$$|k, \mu, k', \mu'\rangle = \sum_j \langle k, \mu, k', \mu' \,|\, j, \mu+\mu'\rangle \,|\, j, \mu+\mu', n, l\rangle \tag{3.5}$$

and we have the following orthogonality relation

$$\sum_{\substack{\mu \\ \mu+\mu'=m}} \langle k, \mu, k', \mu' \,|\, j, m\rangle \langle k, \mu, k', \mu' \,|\, j', m\rangle = \delta_{jj'}. \tag{3.6}$$

In the expression for $P_o\,|\,k, \mu, k', \mu'\rangle$ in (2.3) we insert (3.5), multiply both sides by $\langle k, \mu, k', \mu' \,|\, j_0, m\rangle$, sum over j, use (3.6) and the following four relations

$$\begin{aligned}&\sqrt{(k+\mu+1)(k'-\mu'+1)}\,\langle k+\tfrac{1}{2}, \mu+\tfrac{1}{2}, k'+\tfrac{1}{2}, \mu'-\tfrac{1}{2} \,|\, j, m\rangle \\ &\qquad -\sqrt{(k-\mu+1)(k'+v'+1)}\,\langle k+\tfrac{1}{2}, \mu-\tfrac{1}{2}, k'+\tfrac{1}{2}, \mu'-\tfrac{1}{2} \,|\, j, m\rangle \\ &= \sqrt{(k+k'+j+2)(k+k'-j+1)}\,\langle k, \mu, k', \mu' \,|\, j, m\rangle.\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\sqrt{(k-\mu)(k'-\mu'+1)}\,\langle k-\tfrac{1}{2}, \mu+\tfrac{1}{2}, k'+\tfrac{1}{2}, \mu'-\tfrac{1}{2} \,|\, j, m\rangle \\ &\qquad +\sqrt{(k+\mu)(k'+v'+1)}\,\langle k-\tfrac{1}{2}, \mu-\tfrac{1}{2}, k'+\tfrac{1}{2}, \mu'+\tfrac{1}{2} \,|\, j, m\rangle \\ &= \sqrt{(j+k'-k+1)(j+k-k')}\,\langle k, \mu, k', \mu' \,|\, j, m\rangle.\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\sqrt{(k+\mu+1)(k'+\mu')}\,\langle k+\tfrac{1}{2}, \mu+\tfrac{1}{2}, k'-\tfrac{1}{2}, \mu'-\tfrac{1}{2} \,|\, j, m\rangle \\ &\qquad +\sqrt{(k-\mu+1)(k'-\mu')}\,\langle k+\tfrac{1}{2}, \mu-\tfrac{1}{2}, k'-\tfrac{1}{2}, \mu+\tfrac{1}{2} \,|\, j, m\rangle \\ &= \sqrt{(j+k-k'+1)(j+k'-k)}\,\langle k, \mu, k', \mu' \,|\, j, m\rangle.\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\sqrt{(k-\mu)(k'+\mu')}\,\langle k-\tfrac{1}{2}, \mu+\tfrac{1}{2}, k'-\tfrac{1}{2}, \mu'-\tfrac{1}{2} \,|\, j, m\rangle \\ &\qquad -\sqrt{(k+\mu)(k'-\mu')}\,\langle k-\tfrac{1}{2}, \mu-\tfrac{1}{2}, k'-\tfrac{1}{2}, \mu'+\tfrac{1}{2} \,|\, j, m\rangle \\ &= -\sqrt{(k+k'+j+1)(k+k'-j)}\,\langle k, \mu, k', \mu' \,|\, j, m\rangle.\end{aligned} \tag{3.10}$$

The resulting expression for $P_o\,|\,j, m, n, l\rangle$ is

$$\begin{aligned}P_o\,|\,j, m, l, n\rangle &= \frac{a_{l,n}}{2}\sqrt{(l+j+1)(l-j)}\,|\,j, m, n, l+1\rangle \\ &\quad + \frac{b_{l,n}}{2}\sqrt{(j-n)(j+n+1)}\,|\,j, m, n+1, l\rangle \\ &\quad - \frac{c_{l,n}}{2}\sqrt{(j+n)(j-n+1)}\,|\,j, m, n-1, l\rangle \\ &\quad + \frac{d_{l,n}}{2}\sqrt{(l+j)(l-j-1)}\,|\,j, m, n, l-1\rangle.\end{aligned} \tag{3.11}$$

To prove that the relations (3.7)–(3.10) are valid, one can use e.g. Racahs symmetrical expression for the C–G-coefficients [6]. (Obsiously the relations (3.7)–(3.10) can also be formulated and proved in terms of relations for the 9–j symbols.) From (3.2) it follows that

$$
\begin{aligned}
&N_3\,|j,m,n,l\rangle \\
&= i\Big\{\beta(j+1,m)\big[a_{l,n}\sqrt{((j+1)^2-n^2)(l+j+2)(l+j+1)}\,|j+1,m,n,l+1\rangle \\
&\qquad - b_{l,n}\sqrt{(l^2-(j+1)^2)(j+2+n)(j+1+n)}\,|j+1,m,n+1,l\rangle \\
&\qquad + c_{l,n}\sqrt{(l^2-(j+1)^2)(j+2-n)(j+1-n)}\,|j+1,m,n-1,l\rangle \\
&\qquad - d_{l,n}\sqrt{((j+1)^2-n^2)(l-j-1)(l-j-2)}\,|j+1,m,n,l-1\rangle\big] \\
&\quad \frac{m}{2j(j+1)}\big[a_{l,n}\cdot n\sqrt{(l+j+1)(l-j)}\,|j,m,n,l+1\rangle \\
&\qquad + b_{l,n}\cdot l\cdot\sqrt{(j-n)(j+n+1)}\,|j,m,n+1,l\rangle \\
&\qquad + c_{l,n}\cdot l\cdot\sqrt{(j+n)(j-n+1)}\,|j,m,n-1,l\rangle \\
&\qquad - d_{l,n}\cdot n\cdot\sqrt{(l+j)(l-j-1)}\,|j,m,n,l-1\rangle\big] \\
&\quad + \beta(j,m)\big[a_{l,n}\sqrt{(j^2-n^2)(l+1-j)(l-j)}\,|j-1,m,n,l+1\rangle \\
&\qquad + b_{l,n}\sqrt{(l^2-j^2)(j-n-1)(j-n)}\,|j-1,m,n+1,l\rangle \\
&\qquad - c_{l,n}\sqrt{(l^2-j^2)(j+n+1)(j+n)}\,|j-1,m,n-1,l\rangle \\
&\qquad - d_{l,n}\sqrt{(j^2\,n^2)(l+j)(l+j-1)}\,|j-1,m,n,l-1\rangle\big]\Big\},
\end{aligned}
$$

where
$$\beta(j,m)=\frac{1}{2j}\sqrt{\frac{(j-m)(j+m)}{(2j+1)(2j-1)}}. \tag{3.12}$$

The expressions for $N_\pm\,|j,m,n,l\rangle$ are very similar to (3.12). The changes are

(*a*) in the expression for $N_+\,|j,m,n,l\rangle$: m is replaced by $(m+1)$ in the vectors and

$\beta(j+1,m)$ is replaced by $\gamma(j+1,m+1)$,

$\beta(j,m)$ is replaced by $-\gamma(j,-m)$,

$\dfrac{m}{2j(j+1)}$ is replaced by $\dfrac{\sqrt{(j-m)(j+m+1)}}{2j(j+1)}$,

where
$$\gamma(j,m)=\frac{1}{2j}\sqrt{\frac{(j+m)(j+m-1)}{(2j+1)(2j-1)}} \tag{3.13}$$

(*b*) in the expression for $N_-|j, m, n, l\rangle$: m is replaced by $(m-1)$ in the vectors and

$$\beta(j+1, m) \text{ is replaced by } -\gamma(j+1, -m+1),$$

$$\beta(j, m) \text{ is replaced by } \gamma(j, m),$$

$$\frac{m}{2j(j+1)} \text{ is replaced by } \frac{\sqrt{(j+m)(j-m+1)}}{2j(j+1)}. \tag{3.14}$$

IV. The contraction of the representations

We briefly review the classification of the unitary irreducible representations of $P(1,3)$ [7], and establish a notation for these. We introduce

ε = sign of the energy,

λ_1 = helicity,

ε_1 = sign of the eigenvalues of the operator corresponding to the rotation in the little group in the case $\omega < 0$.

ε, λ_1 and ε_1 are invariants for special representations. The notation for the representations will be the following:

The invariants and their values	Notation for the representations
$\omega = \mu^2 > 0,\ \omega' = -\mu^2 s(s+1)$, where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots;\ \varepsilon = \pm 1$.	$P(\mu^2, s, \varepsilon)$.
$\omega = 0,\ \omega' = -a^2 < 0,\ \varepsilon = \pm 1$.	$P(0, a^2, \varepsilon)$.
$\omega = 0,\ \omega' = 0,\ \varepsilon = \pm 1$, $\lambda_1 = 0,\ \pm\frac{1}{2}, \pm 1, \ldots$	$P(0, 0, \varepsilon, \lambda_1)$.
$\omega = -\mu^2 < 0,\ \omega' = -\mu^2 \alpha$, $\alpha > 0$ for single valued representations, $\alpha > \frac{1}{4}$ for double-valued representations.	$P(-\mu^2, \alpha)$.
$\omega = -\mu^2,\ \omega' = \mu^2 s(s+1)$, where $s = -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots;\ \varepsilon_1 = \pm 1$.	$P(-\mu^2, s, \varepsilon_1)$.

Sometimes we only want to distinguish the representations with respect to their values of ω. For the cases $\omega > 0, \omega = 0$ and $\omega < 0$ we then use the abbreviated notation $P(\mu^2)$, $P(0)$ and $P(-\mu^2)$ respectively.

In order to be able to construct the unitary representations of $P(1,3)$ we will now consider limits of special sets of the representations of $L(1,4)$. The different limits which are to be considered will be specified below.

1. *Contraction to representations $P(\mu^2)$*

Consider the representations $\nu_{r,\sigma}$. We amplify the notation for the basis vectors in the representation space for this case and write them as $|j, m, n, l;\ \sigma, r\rangle$. In all the three cases 1a, 1b and 1c of section II the possible values of σ have only a lower bound and we can therefore consider the following limit

$$\eta \to 0,\ \sigma \to \infty \quad \text{so that} \quad \lambda^2 \sigma = \mu^2. \tag{4.1}$$

We further put $r = s$. Then

$$\lim_{(4,1)} \lambda^2 \nu_{s,\sigma}(\Omega) = \mu^2,$$

$$\lim_{(4,1)} \lambda^2 \nu_{s,\sigma}(\Omega') = -\mu^2 s(s+1).$$

To get a faithful representation of $P(1,3)$, and in particular to get a representation of the type $P(\mu^2)$, we must, together with the limit (4.1) consider a sequence of basis vectors in the representation space, where l increases as λ approaches zero. We consider the limit

$$l \to \infty \quad \text{so that} \quad \lambda \cdot l = p \tag{4.3}$$

and we write $\quad |j, m, n, l;\ \sigma, r\rangle \to |j, m, n, p;\ \mu^2, s\rangle.$

If the limits (4.1) and (4.3) are taken in the expressions for $a_{l,n}$, $b_{l,n}$, $c_{l,n}$ and $d_{l,n}$ we get

$$\lim_{(4.1)(4.3)} \lambda P_o |j, m, l;\ \sigma, s\rangle = \sqrt{p^2+\mu^2}\, |j, m, n, p;\ \mu^2 s\rangle,$$

i.e. we have formally

$$p_0 |j, m, n, p;\ \mu^2, s\rangle = \sqrt{p^2+\mu^2}\, |j, m, n, p;\ \mu^2, s\rangle. \tag{4.4}$$

For p_3 and $p_\pm$ we obtain from (3.1):

$$\begin{aligned}
p_3 &|j, m, n, p;\ \mu^2, s\rangle \\
&= -\frac{p}{(j+1)} \sqrt{\frac{((j+1)^2-m^2)((j+1)^2-n^2)}{(2j+3)(2j+1)}}\, |j+1, m, n, p;\ \mu^2, s\rangle \\
&\quad + \frac{m \cdot n \cdot p}{j(j+1)} |j, m, n, p;\ \mu^2, s\rangle - \frac{p}{j} \sqrt{\frac{(j^2-m^2)(j^2-n^2)}{(2j+1)(2j-1)}}\, |j-1, m, n, p;\ \mu^2, s\rangle, \\
p_+ &|j, m, n, p;\ \mu^2, s\rangle \\
&= \frac{p}{(j+1)} \sqrt{\frac{(j+m+1)(j+m+2)((j+1)^2-n^2)}{(2j+3)(2j+1)}}\, |j+1, m+1, n, p;\ \mu^2, s\rangle \\
&\quad + \frac{n \cdot p}{j(j+1)} \sqrt{(j-m)(j+m+1)}\, |j, m+1, n, p;\ \mu^2, s\rangle \\
&\quad - \frac{p}{j} \sqrt{\frac{(j-m)(j-m-1)(j^2-n^2)}{(2j+1)(2j-1)}}\, |j-1, m+1, n, p;\ \mu^2, s\rangle,
\end{aligned}$$

$$P_-|j,m,n,p;\mu^2,s\rangle$$
$$=-\frac{p}{(j+1)}\sqrt{\frac{(j-m+2)(j-m+1)((j+1)^2-n^2)}{(2j+3)(2j+1)}}\,|j+1,m-1,n,p;\mu^2,s\rangle$$
$$+\frac{n\cdot p}{j(j+1)}\sqrt{(j+m)(j-m+1)}\,|j,m-1,n,p;\mu^2,s\rangle$$
$$+\frac{p}{j}\sqrt{\frac{(j+m)(j+m-1)(j^2-n^2)}{(2j+1)(2j-1)}}\,|j-1,m-1,n,p;\mu^2,s\rangle. \quad (4.5)$$

$\mathbf{M}$ and $\mathbf{p}$ form the Lie algebra of $E(3)$. $\mathbf{p}^2$ and $\mathbf{p}\cdot\mathbf{M}$ are invariants of $E(3)$ and in the basis $|j,m,n,p;\mu^2,s\rangle$ we have

$$\left.\begin{aligned}\mathbf{p}^2|j,m,p;\mu^2 s\rangle&=p^2|j,m,n,p;\mu^2,s\rangle,\\ \mathbf{p}\cdot\mathbf{M}|j,m,n,p;\mu^2,s\rangle&=n\cdot p\,|j,m,p;\mu^2,s\rangle.\end{aligned}\right\} \quad (4.6)$$

For fixed values of n and p the basis vectors $|j,m,n,p;\mu^2,s\rangle$ with

$$m=-j,-j+1,\dots j-1,j,\ j=|n|,|n|+1,|n|+2\dots,\ p>0$$

span a representation space for an irreducible representation of $E(3)$.

Next we apply the limits (4, 1) and (4, 3) to the expression for $N_3|j,m,n,l\rangle$ in (3, 14). Consider the terms in which j, m and n are not changed. Apart from an unessential factor these terms are

$$F=a_{l,n}\sqrt{(l-j)(l+j+1)}\,|j,m,n,l+1;\mu^2,s\rangle$$
$$-d_{l,n}\sqrt{(l+j)(l-j-1)}\,|j,m,n,l-1;\mu^2,s\rangle. \quad (4.7)$$

We have $\lambda l=p$ i.e. $\lambda(l\pm1)=p\pm\lambda$. If we suppress all indices but p in the basis vectors and insert the values of $a_{l,n}$ and $d_{l,n}$ into (4, 7) we can write

$$\lim_{(4.1)(4.3)} F=\lim_{\lambda\to0}\left\{\sqrt{p^2+\mu^2+\lambda p}\,|p+\lambda\rangle-\sqrt{p^2+\mu^2-\lambda p}\,|p-\lambda\rangle\right\}$$
$$=2\sqrt{p^2+\mu^2}\frac{\partial}{\partial p}|p\rangle+\frac{p}{\sqrt{p^2+\mu^2}}|p\rangle=F_0.$$

The expression for F_0 is somewhat simplified if we introduce the (positive) variable $E=\sqrt{p^2+\mu^2}$ and change to basis vectors

$$|E\rangle=\sqrt{E}\,|p\rangle.$$

We then have
$$F_0=2p\frac{\partial}{\partial E}|E\rangle.$$

Obviously the new vectors $|E\rangle$ can be introduced into (4.4) and (4.5) without changing any of the coeffients in the equations.

Differences analogous to (4.7) will occur also for the terms which have the j-values $j \pm 1$. Further we have

$$\lim_{(4.1)(4.3)} b_{l,n} \cdot l \cdot \sqrt{(j-n)(j+n+1)}\,|j, m, n+1, l; \sigma, s\rangle$$

$$= \frac{\mu}{p}\sqrt{(s-n)(s+n+1)(j-n)(j+n+1)}\,|j, m, n+1, p; \mu^2, s\rangle. \tag{4.8}$$

The factors in front of $|j, m, n-1, p; \mu^2, s\rangle$ and $|j \pm 1, m, n \pm 1, p; \mu^2, s\rangle$ are of the same structure. Collecting the results we have in terms of the basis vectors $|j, m, n, E; \mu^2, s\rangle$:

$$\begin{aligned}
&N_3\,|j, m, n, E; \mu^2 s\rangle \\
&= i\Big\{\beta(j+1, m)\Big[\sqrt{(j+1)^2-n^2}\left(\frac{2(j+1)}{p}E + 2p\frac{\partial}{\partial E}\right)|j+1, m, n, E; \mu^2, s\rangle \\
&\quad -\frac{\mu}{p}\sqrt{(s-n)(s+n+1)(j+n+2)(j+n+1)}\,|j+1, m, n+1, E; \mu^2, s\rangle \\
&\quad -\frac{\mu}{p}\sqrt{(s+n)(s-n+1)(j-n+2)(j-n+1)}\,|j+1, m, n-1, E; \mu^2, s\rangle\Big] \\
&\quad -\frac{m}{2j(j+1)}\Big[2np\frac{\partial}{\partial E}\,|j, m, n, E; \mu^2, s\rangle \\
&\quad +\frac{\mu}{p}\sqrt{(s-n)(s+n+1)(j-n)(j+n+1)}\,|j, m, n+1, E; \mu^2, s\rangle \\
&\quad -\frac{\mu}{p}\sqrt{(s+n)(s-n+1)(j+n)(j-n+1)}\,|j, m, n-1, E; \mu^2, s\rangle\Big] \\
&\quad +\beta(j, m)\Big[\sqrt{j^2-n^2}\left(-\frac{2j}{p}E + 2p\frac{\partial}{\partial E}\right)|j-1, m, n, E; \mu^2, s\rangle \\
&\quad +\frac{\mu}{p}\sqrt{(s-n)(s+n+1)(j-n-1)(j-n)}\,|j-1, m, n+1, E; \mu^2 s\rangle \\
&\quad +\frac{\mu}{p}\sqrt{(s+n)(s-n+1)(j+n-1)(j+n)}\,|j-1, m, n-1, E; \mu^2, s\rangle\Big]. \qquad (4.9)
\end{aligned}$$

The expressions for $N_\pm\,|j, m, n, E; \mu,^2, s\rangle$ are again obtained with the help of the substitutions (3.13) and (3.14). Thereby the action of all the operators $\mathbf{M}, \mathbf{N}, \mathfrak{p}$ and p_0 in the basis $|j, m, n, E; \mu^2, s\rangle$ is known. These relations have been obtained in a very formal way. We must further specify the normalization of the basis vectors before we can study the commutation relations of the operators. Basis vectors $|j, m, n, l\rangle$ corresponding to different values in any one of the indices are orthogonal and we have

$$\sum_{j', m', n', l'} \langle j', m', n', l' \mid j, m, n, l\rangle = 1. \tag{4.10}$$

If we assume that λ is a positive parameter, p is also positive, i.e.

$$p = +\sqrt{E^2 - \mu^2}.$$

After the introduction of the continuous variable p it is therefore natural to assume that the vectors $|j, m, n, p; \mu^2, s\rangle$ are orthogonal in the discrete indices j, m and n and that we have

$$\sum_{j', m', n'} \int_0^\infty \langle j', m', n', p' \mid j, m, n, p\rangle \, dp = 1$$

and consequently

$$\int_0^\infty \frac{dp}{E} \langle j, m, n, E' \mid j, m, n, E\rangle = \int_\mu^\infty \frac{dE}{p} \langle j, m, n, E' \mid j, m, n, E\rangle = 1, \tag{4.11}$$

i.e.

$$\langle j, m, n, E' \mid j, m, n, E\rangle = p \cdot \delta(E' - E). \tag{4.12}$$

It is now easy to check that with the assumption (4.12) the operators $\mathbf{M}, \mathbf{N}, \mathbf{p}$ and p_0 are hermitian, satisfy the correct commutation relations and give

$$\omega = \mu^2, \omega' = -\mu^2 s(s+1),$$

We have introduced the positive energy variable $E = +\sqrt{p^2 + \mu^2}$ and therefore we have constructed the infinitesimal operators for a representation $P(\mu^2, s, +1)$. (The irreducibility follows from the general theory of the unitary representations of $P(1,3)$ and can also be seen from an inspection of the formulas.) The representation space is spanned by vectors $|j, m, n, E; \mu^2, s\rangle$, where n assumes the values $-s, \ -s+1, \dots s-1, s$, and where $j = |n|, |n|+1, \ |n|+2, \dots$ and $m = -j, -j+1, \dots j-1, j$. The values of E are restricted to $\mu < E < \infty \, (\mu > 0)$.

In checking the hermiticity conditions, the commutation relations etc. one never uses the fact that E is positive, only the connection $EdE = pdp$ is used, which, however, is valid for $E = \pm\sqrt{p^2 + \mu^2}$. This means that only by prescribing that E shall be negative, $-\infty < E < -\mu$, we achieve that the formulas for $\mathbf{M}, \mathbf{N}, \mathbf{p}$ and p_0, as given above, define a representation $P(\mu^2, s, -1)$.

We do not intend to treat here the problem of an explicit realization of the representation space, the existence of which we assume. Realizations of representation spaces for unitary representations of $E(3)$ are known and the problem is then to define the action of time translations and Lorentz transformations in a superposition of such representation spaces. In terms of the unitary representations of $E(3)$ the result of this subsection can also be formulated in the following way: if a representation $P(\mu^2, s, \varepsilon)$ is reduced with respect to representations of $E(3)$ one gets a direct integral in p from zero to infinity of representations (p, n) of $E(3)$ and the values of n which occur are $-s, -s+1, -\dots s-1, s$. (Cf. the decomposition of the identity

$$I = \sum_{j, m, n} \int_0^\infty dp \, | j, m, n, p\rangle \langle j, m, n, p |.)$$

2. *Contraction to representations* $P(0)$

Consider the representations $\nu_{r,\sigma}$ and the limit

$$\lambda\to 0,\ r\to\infty,\ \sigma\to\infty,\ l\to\infty \quad \text{so that} \quad \lambda\sigma=a,\ \lambda r^2=a,\ \lambda l=p. \tag{4.13}$$

in the representations and in the basis vectors. Then

$$\left.\begin{aligned} &\lim_{(4.13)} \lambda^2\,\nu_{r,\sigma}(\Omega)=0,\\ &\lim_{(4.13)} \lambda^2\,\nu_{r,\sigma}(\Omega')=-a^2. \end{aligned}\right\} \tag{4.14}$$

When $\omega=0, \omega'\neq 0, \varepsilon$ is also an invariant. We introduce again the energy variable, $p=\sqrt{E^2}$ and for $\varepsilon=+1$ we have $p=E$. In analogy with the former case we introduce basis vectors $|j,m,n;E,0,-a^2\rangle$. Using (4.12) we then find for the case $\varepsilon=+1$:

$$p_0\,|j,m,n,E;\,0,-a^2\rangle = E\,|j,m,n,E;\,0,-a^2\rangle. \tag{4.15}$$

The expressions for $p_3\,|j,m,n,E;\,0,-a^2\rangle$ and $p_\pm\,|j,m,n,E;\,0,-a^2\rangle$ are still given by (4.5) after the appropriate change in the notation for the basis vectors. However, we prefer to retain the letter p in the coefficients of this equation. For $N_3\,|j,m,n,E;\,0,-a^2\rangle$ we find (and we choose to use both letters E and p due to the fact that we will below consider the extension to negative values of E; our reason for retaining p is that we want to write the formulas for this case, $\omega=0$, in a form which is as similar as possible to the earlier one). Applying the limit (4.13) to (3.14) we then find: (we assume $a>0$)

$$\begin{aligned}
&N_3\,|j,m,n,E;\,0,-a^2\rangle\\
&\quad = i\left\{\beta(j+1,m)\left[\sqrt{(j+1)^2-n^2}\left(\frac{2(j+1)}{p}\,E+2p\,\frac{\partial}{\partial E}\right)|j+1,m,n,E;0,-a^2\rangle\right.\right.\\
&\qquad\qquad -\frac{a}{p}\sqrt{(j+n+2)(j+n+1)}\,|j+1,m,n+1,E;\,0,-a^2\rangle\\
&\qquad\qquad \left.-\frac{a}{p}\sqrt{(j-n+2)(j-n+1)}\,|j+1,m,n-1,E;\,0,-a^2\rangle\right]\\
&\qquad -\frac{m}{2j(j+1)}\left[2nE\,\frac{\partial}{\partial E}\,|j,m,n,E;\,0,-a^2\rangle\right.\\
&\qquad\qquad +\frac{a}{p}\sqrt{(j-n)(j+n+1)}\,|j,m,n+1,E;\,0,-a^2\rangle\\
&\qquad\qquad \left.-\frac{a}{p}\sqrt{(j+n)(j-n+1)}\,|j,m,n-1,E;\,0,-a^2\rangle\right]\\
&\qquad +\beta(j,m)\left[\sqrt{j^2-n^2}\left(-\frac{2j}{p}\,E+2p\,\frac{\partial}{\partial E}\right)|j-1,m,n,E;0,-a^2\rangle\right.
\end{aligned}$$

$$+\frac{a}{p}\sqrt{(j-n-1)(j-n)}\,|j-1,m,n+1,E;\ 0,\ -a^2\rangle$$
$$\left.\left.+\frac{a}{p}\sqrt{(j+n-1)(j+n)}\,|j-1,m,n-1,E;\ 0,\ -a\rangle\right]\right\}. \tag{4.16}$$

The expressions for $N_{\pm}|j,m,E;\ 0,-a^2\rangle$ are obtained by means of (3.13) and (3.14). We assume that $|j,m,n,E;\ 0,a^2\rangle$ are normalized according to (4.12). We can now check that the operators $\mathbf{M}$, $\mathbf{N}$, $\mathbf{p}$ and p_0 defined in (4.15), (4.5) and (4.16) (combined with (3.13) and (3.14)) are hermitian, satisfy the correct commutation relations and give $\omega=0$, $\omega'=-a^2$. We have further assumed $\varepsilon=+1$ i.e. we have constructed the infinitesimal operators for a representation $P(0,-a^2,+1)$. The representation space is spanned by vectors $|j,m,n,E;\ 0,-a^2\rangle$ where $n=0,\pm1,\pm2$ or $n=\pm\frac{1}{2},\pm\frac{3}{2}\ldots$ and where for each n the values of j and m are $j=|n|,|n|+1,|n|+2,\ldots,m=-j,-j+1,\ldots j-1,j$. Further $0<E<\infty$. As is easily checked we get a representation $P(0,-a^2,-1)$ by prescribing that E shall take negative values: $-\infty<E<0$.

The representations $P(0,0,\varepsilon,\lambda_1)$ are obtained by putting $a=0$ and $n=\lambda_1$ in the formulas of this subsection. They are also obtained by putting $\mu=0$ in the formulas of subsection (IV, 1).

In terms of the unitary representations of $E(3)$ the result of this subsection can be formulated in the following way: (i) if a representation $P(0,-a^2,\varepsilon)$ is reduced with respect to the representations (p,n) of $E(3)$ one gets a direct integral from zero to infinity in p and the values of n which occur are for one-valued representations $n=0,\pm1,\pm2\ldots$ and for double-valued representations $n=\pm\frac{1}{2},\pm\frac{3}{2}\ldots$ (ii) if a representation $P(0,0,\varepsilon,\lambda_1)$ is reduced with respect to the representations (p,n) of E_3 one gets a direct integral from zero to infinity in p and the only n-value which occurs is $n=\lambda_1$.

3. *Contraction to representations* $P(-\mu^2)$

For both cases $P(-\mu^2,\alpha)$ and $P(-\mu^2,s,\varepsilon_1)$ we will consider the following limit

$$\lambda\to0,\ r\to\infty,\ l\to\infty \quad \text{so that} \quad \lambda l=p,\ \lambda r=\mu \ \ (\text{thus} \quad p^2\geqslant\mu^2). \tag{4.17}$$

Then
$$\lim_{(4.17)}\lambda P_0|j,m,n,l\rangle=\sqrt{p^2-\mu^2}\,|j,m,n,p\rangle$$

is valid both for representations of the continuous class and the discrete class (cf. section II). However, when $\omega<0$, then sign of the "energy" E in $E^2=p^2-\mu^2$ is not an invariant and we must therefore in this case introduce from the beginning the condition that the eigenvalue E of p_0 can take both positive and negative values and we write

$$p_0|j,m,n,E;\ -\mu^2\rangle=E\,|j,m,n,E;\ -\mu^2\rangle, \tag{4.18}$$

where we assume that the vectors $|j,m,n,E;-\mu^2\rangle$ are normalized according to

$$\int_{-\infty}^{\infty} \frac{dE}{p} \langle j, m, n, E'; \ -\mu^2 \,|\, j, m, n, E; \ -\mu^2\rangle = 1. \tag{4.19}$$

In case (a) below we will construct representations $P(-\mu^2, \alpha)$ by considering limits of representations $\nu_{r,\sigma}$ and in case (b) we construct representations $P(-\mu^2, s, \varepsilon_1)$ by considering the limits of representations $\pi^{\varepsilon_1}_{r,q}$

(a) Contraction to representations $P(-\mu^2, \alpha)$.

We note that for $r>0$ the domain of variation of σ for single-valued and double-valued representations respectively is the same as that of α in $P(-\mu^2, \alpha)$. Therefore we put $\sigma = \alpha$ and consider the limit (4.17). We find

$$\left.\begin{aligned} &\lim_{(4.17)} \lambda^2 \nu_{r,\alpha}(\Omega) = -\mu^2, \\ &\lim_{(4.17)} \lambda^2 \nu_{r,\alpha}(\Omega) = -\mu^2 \alpha. \end{aligned}\right\} \tag{4,20}$$

The basis vectors are denoted by $|j, m, n, E; -\mu^2, \alpha\rangle$. The expression for $N_3 \,|\, j, m, n, E; -\mu^2, \alpha\rangle$ differs from (4.9) only in the following respect:

$$\left.\begin{aligned} &\sqrt{(s-n)(s+n+1)} \quad \text{is replaced by} \quad \sqrt{n(n+1)+\alpha}, \\ &\sqrt{(s+n)(s-n+1)} \quad \text{is replaced by} \quad \sqrt{n(n-1)+\alpha}, \end{aligned}\right\} \tag{4.21}$$

(these factors occur in front of $|j \ldots n \pm 1 \ldots\rangle$ and $|j \pm 1 \ldots n \pm 1, \ldots\rangle$). (4.5) and (4.18) give the expression for $\mathbf{p}$ and p_0 and one can now check that the operators $\mathbf{M}, \mathbf{N}, \mathbf{p}$ and p_0 whose action in the basis $|j, m, n, E; -\mu^2, \alpha\rangle$ it thereby given, define a representation $P(-\mu^2, \alpha)$ of $P(1,3)$. The representation space is spanned by vectors $|j, m, n, E; -\mu^2, \alpha\rangle$ where $-\infty < E < \infty$, $n = 0, \pm 1, \pm 2$ for one-valued representations and $n = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$ for two-valued representations. For each n, j assumes the values $j = |n|, |n|+1, \ldots$ and $m = -j, -j+1, \ldots j-1, j$.

(b) Contraction to representations $P(-\mu^2, s, \varepsilon_1)$.

Put $q - 1 = s$. Then $s = -\frac{1}{2}, 0, \frac{1}{2} \ldots$ for the representations $\pi^{\pm}_{r,s+1}$ and we find

$$\left.\begin{aligned} &\lim_{(4.17)} \lambda^2 \pi^{\pm}_{r,s+1}(\Omega) \;=\; -\mu', \\ &\lim_{(4.17)} \lambda^2 \pi^{\pm}_{r,s+1}(\Omega') = \mu^2 s(s+1). \end{aligned}\right\} \tag{4.22}$$

In $\pi^{+}_{r,s+1}$ we have $n \geqslant q = s+1$ and min $q = 1$ or $\frac{1}{2}$ and in $\pi^{-}_{r,s+1}$ we have $n \leqslant -q = -s-1$ and min $q = 1$ or $\frac{1}{2}$. Denote the basis vectors by

$$|j, m, n, E; -\mu^2, s, \varepsilon_1\rangle.$$

By applying the limit (4.17) to the infinitesimal relations for the representations $\pi^{\varepsilon_1}_{r,s+1}$ we obtain the infinitesimal relations for $P(-\mu^2, s, \varepsilon_1)$. The expressions for $\mathbf{p}$ and p_0 are the same as before and the expression for

$$N_3 \,|\, j, m, n, E; -\mu^2, s, \varepsilon_1\rangle$$

differs from (4.9) only in the following respect

$$\left.\begin{array}{l} \sqrt{(s-n)(s+n+1)} \quad \text{is replaced by} \quad \sqrt{(n-s)(n+s+1)} \\ \sqrt{(s+n)(s-n+1)} \quad \text{is replaced by} \quad \sqrt{(n+s)(n-s-1)}. \end{array}\right\} \tag{4.23}$$

The representation space is spanned by vectors $|j, m, n, E; -\mu^2, s, \varepsilon_1\rangle$ where $-\infty < E < \infty$, $n = s+1, s+2, \ldots$ for $\varepsilon_1 = +1$ and $n = -s-1, -s-2, \ldots$ for $\varepsilon_1 = -1$. As before we have for each n the j-values $j = |n|, |n|+1, \ldots$ and $m = -j, -j+1, \ldots j-1, j$.

Herewith we have completed the construction of the infinitesimal operators $\mathbf{M}, \mathbf{N}, \mathbf{p}$ and p_0 of $P(1,3)$ in the chosen angular momentum basis. In order to obtain all the unitary irreducible representations of $P(1.3)$ it was in some cases necessary to give certain prescriptions not inherent in the process of contraction (in which we always kept $\lambda > 0$). Some of the results obtained above, namely those concerning the representations $P(\mu^2, s, \varepsilon)$ and $P(0, 0, \varepsilon, \lambda_1)$ have also been derived by J. S. Lomont and H. E. Moses [8]. These authors use the commutation relations of $\mathbf{M}, \mathbf{N}, \mathbf{p}$ and p_0 in a direct way by computing "reduced matrix elements".

Institute of Theoretical Physics, Göteborg, March 1965

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Tryckt den 30 oktober 1965

Uppsala 1965. Almqvist & Wiksells Boktryckeri AB

Arkiv för Fysik

ARKIV FÖR FYSIK

UTGIVET AV
KUNGL. SVENSKA VETENSKAPSAKADEMIEN
(Band 31 nr 34)

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On the unitary irreducible representations of the (1+4) de Sitter group

ALMQVIST & WIKSELL
STOCKHOLM
GÖTEBORG · UPPSALA
1966

ARKIV FÖR FYSIK Band 31 nr 34

3 65 14 2 Communicated 10 November 1965 by L. Hulthén and N. Svartholm

On the unitary irreducible representations of the (1+4) de Sitter group

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ABSTRACT

The unitary irreducible representations of the covering group of the $(1+4)$ de Sitter group are determined using a method based on infinitesimal transformations in the group spaces of the first two subgroups K and A in the canonical decomposition $G = K \cdot A \cdot N$ of any semi-simple Lie group.

1. Introduction

In the present paper we will consider the irreducible unitary representations of the the $(1+4)$ de Sitter group which will be denoted by $L(1,4)$ according to the notation in ref. [1]. More precisely, $L(1.4)$ denotes the identity component of the group of real linear homogeneous transformations which leave invariant the quadratic form

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

The group $L(1,4)$ appears in a number of contexts of physical interest. Among these we may mention that it is connected with the Poincaré group by a contraction in the sense of Inönü and Wigner [2]. The limiting procedure inherent in the contraction can also be used to construct unitary representations of the Poincaré group as certain limits of unitary representations of $L(1,4)$ [3]. Recently $L(1,4)$ has also been considered as a possible spectrum generating group of the non-relativistic hydrogen-atom problem [4, 5].

In the physical applications it is the unitary irreducible representations of $L(1,4)$ which are of interest. The classification of these has been treated by Thomas [6], Newton [7] and Dixmier [8], who all used infinitesimal methods. Later the same representations have been determined by Takahashi [9], using Mackey's theory of induced representations. Takahashi thus gets the representations in global form and with one minor exception he also constructs the appropriate scalar products. In the present paper the unitary irreducible representations of $L(1,4)$ are determined with the method described in ref. [1]. As has been shown [1, 11], this method gives the complete classification of the unitary irreducible representations of the homogeneous Lorentz groups in two and three space dimensions. Our reason for treating again the problem of the determination of all the unitary irreducible representations of $L(1,4)$ is that we want to investigate further the power of the method of ref. [1].

This method can be applied to any group $L(p, q)$, i.e. the group of real linear homogeneous transformations which leave invariant the quadratic form

$$x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2,$$

and which are continuously connected to the identity transformation. Starting from the Iwasawa decomposition [12]

$$L(p, q) = K \cdot A \cdot N,$$

where K is the maximal compact subgroup of $L(p, q)$, A Abelian, N nilpotent and invariant in $A \cdot N$, one can see that there are always at least as many variables in the group spaces of K and A as there are labels required to characterize an arbitrary vector in a representation characterized by arbitrary values of the invariants. One can thus use the group spaces of K and A as a carrier space for unitary representations of $L(p, q)$ [1]. The number of labels required and the number of variables in the group spaces of K and A is the same if $q-p$ equals 0,1 or 2 (it is sufficient to consider, for example, $q \geqslant p$) and in these cases the method is particularly simple to apply. It is therefore also of a methodical interest to consider a case where $q-p>2$ and the simplest case is then $L(1,4)$. It turns out that the complication arising from the fact that there is one redundant variable in the carrier space is not serious and the calculations can be performed in a simple and explicit manner, to give all the unitary irreducible representations of $L(1,4)$. In the course of the calculations a specific choice of basis is made in the representation space and for all the different series of representations a positive definite scalar product is constructed.

In section 2 a parameterization of $L(1,4)$, which is connected with the Iwasawa decomposition, is introduced and a corresponding realization of the Lie algebra of $L(1,4)$ is obtained. In section 3 we make an ansatz for the elimination of the redundancy in the degrees of freedom in the representation space. After having introduced the irreducibility condition we obtain in this way an explicit basis and the action of the operators of the Lie algebra in this basis is determined. In section 4 we can, after the introduction of a scalar product in the representation space, formulate the unitarity conditions. The conditions so obtained enable us to enumerate all the different series of irreducible unitary representations of $L(1,4)$.

2. $L(1,4)$ and its Lie algebra

The Lie algebra $\mathrm{l}(1,4)$ of $L(1,4)$ is generated by the real matrices

$$\begin{bmatrix} 0 & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{01} & 0 & -a_{12} & -a_{13} & -a_{14} \\ a_{02} & a_{12} & 0 & -a_{23} & -a_{24} \\ a_{03} & a_{13} & a_{23} & 0 & -a_{34} \\ a_{04} & a_{14} & a_{24} & a_{34} & 0 \end{bmatrix} \tag{2.1}$$

Suitable basis elements $L_{\mu\nu}$ are defined as follows

$$\left.\begin{aligned} L_{\mu\nu} &= -e_{\mu\nu}+e_{\nu\mu} \quad \text{for} \quad \mu,\nu>0, \\ L_{\mu\nu} &= e_{\mu\nu}+e_{\nu\mu} \quad \text{for} \quad \mu=0,\nu>0, \\ L_{\nu\mu} &= -L_{\mu\nu}, \end{aligned}\right\} \tag{2.2}$$

where the matrix $e_{\mu\nu}$ is zero except at the position $(\mu\nu)$ where it takes the value 1. In terms of these basis elements the commutation relations are

$$[L_{\mu\nu}, L_{\varrho\sigma}] = g_{\nu\varrho}\, L_{\mu\sigma} + g_{\mu\sigma}\, L_{\nu\varrho} - g_{\mu\varrho}\, L_{\nu\sigma} - g_{\nu\sigma}\, L_{\mu\varrho}, \tag{2.3}$$

where $g_{\mu\nu}$ is the metric tensor

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & -1 \end{bmatrix}. \tag{2.4}$$

The generators $L_{\mu\nu}$ for $\mu,\nu>0$ span the Lie algebra of the rotation group in four dimensions, $SO(4)$. This subgroup is also the maximal compact subgroup of $L(1,4)$. The generators $L_{0\nu}$ correspond to "accelerations" in the directions x_ν.

The invariants of the group $L(1,4)$ or more precisely a basis for the centre of the universal enveloping algebra can be chosen as follows

$$\begin{aligned} I_2 &= L_{12}^2 + L_{13}^2 + L_{14}^2 + L_{23}^2 + L_{24}^2 + L_{34}^2 - L_{01}^2 - L_{02}^2 - L_{03}^2 - L_{04}^2, \\ I_4 &= (L_{12}\,L_{34} - L_{13}\,L_{24} + L_{14}\,L_{23})^2 - (L_{02}\,L_{34} - L_{03}\,L_{24} + L_{04}\,L_{23})^2 \\ &\quad - (L_{01}\,L_{34} - L_{03}\,L_{14} + L_{04}\,L_{13})^2 - (L_{01}\,L_{24} - L_{02}\,L_{14} + L_{04}\,L_{12})^2 \\ &\quad - (L_{01}\,L_{23} - L_{02}\,L_{13} + L_{03}\,L_{12})^2. \end{aligned} \tag{2.5}$$

In an irreducible unitary representation of $L(1,4)$ I_2 and I_4 have to take constant real values.

We will now give a realization of the Lie algebra $\mathfrak{l}(1,4)$ by means of differential operators on a carrier space. This carrier space is the product space of the group spaces of K and A in the decomposition [12]

$$L = K \cdot A \cdot N \tag{2.6}$$

of any sem-simple Lie group into three subgroups. In the case of $L = L(1,4)$, K is the maximal compact subgroup, generated by $L_{\mu\nu}$ for $\mu,\nu>0$. The Abelian subgroup A is generated by L_{01} while the nil-potent (and here in fact also Abelian) subgroup N is generated by $L_{02}-L_{12}$, $L_{03}-L_{13}$ and $L_{04}-L_{14}$. Note also that A and N together form a group and that N is an invariant subgroup in this product. Following reference [1] we now introduce parameters into the subgroups K, A and N. For a group element n of N we get

$$n = e^{r(L_{02}-L_{12})}\, e^{s(L_{03}-L_{13})}\, e^{t(L_{04}-L_{14})}$$

$$= \begin{bmatrix} 1+\frac{r^2}{2}+\frac{s^2}{2}+\frac{t^2}{2} & -\frac{r^2}{2}-\frac{s^2}{2}-\frac{t^2}{2} & r & s & t \\ \frac{r^2}{2}+\frac{s^2}{2}+\frac{t^2}{2} & 1-\frac{r^2}{2}-\frac{s^2}{2}-\frac{t^2}{2} & r & s & t \\ r & -r & 1 & 0 & 0 \\ s & -s & 0 & 1 & 0 \\ t & -t & 0 & 0 & 1 \end{bmatrix} \tag{2.7}$$

while an element a of A becomes

$$a = e^{\lambda L_{01}} = \begin{bmatrix} Ch\,\lambda & Sh\,\lambda & 0 & 0 & 0 \\ Sh\,\lambda & Ch\,\lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.8}$$

The group element k of K will later be parameterized by two sets of Euler angles but for the moment we just write k in the form

$$k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & k_{11} & k_{12} & k_{13} & k_{14} \\ 0 & k_{21} & k_{22} & k_{23} & k_{24} \\ 0 & k_{31} & k_{32} & k_{33} & k_{34} \\ 0 & k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}, \tag{2.9}$$

The matrix representing the general group element $g = k\cdot a\cdot n$ of $L(1,4)$ is now easily obtained from (2.7) – (2.9). By multiplication from the left of this group element by another group element the parameters are changed. If the multiplying element is infinitesimal and of the form

$$e^{-\varepsilon L_{01}} \approx \begin{bmatrix} 1 & -\varepsilon & 0 & 0 & 0 \\ -\varepsilon & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad |\varepsilon| \ll 1, \tag{2.10}$$

one can derive a differential expression for L_{01} in terms of the parameters of K and A [1]. One finds the following expression for the generator L_{01}

$$L_{01} = \sum_{\mu,\nu} (k_{\mu 1} k_{1\nu} - \delta_{\mu 1} \delta_{1\nu}) \frac{\partial}{\partial k_{\mu\nu}} - k_{11} \frac{\partial}{\partial \lambda}. \tag{2.11}$$

We now choose suitable parameters on K by means of the formula

$$k = e^{\alpha_1 L_{34}} e^{\alpha_2 L_{12}} e^{\alpha_3 L_{24}} e^{\alpha_4 L_{13}} e^{\alpha_5 L_{34}} e^{\alpha_6 L_{12}}$$

$$= \begin{bmatrix} C\alpha_2 C\alpha_4 C\alpha_6 - S\alpha_2 C\alpha_3 S\alpha_6; & -C\alpha_2 C\alpha_4 S\alpha_6 - S\alpha_2 C\alpha_3 C\alpha_6; & -C\alpha_2 S\alpha_4 C\alpha_5 + S\alpha_2 S\alpha_3 S\alpha_5; & C\alpha_2 S\alpha_4 S\alpha_5 + S\alpha_2 S\alpha_3 C\alpha_5 \\ C\alpha_2 C\alpha_3 S\alpha_6 + S\alpha_2 C\alpha_4 C\alpha_6; & C\alpha_2 C\alpha_3 C\alpha_6 - S\alpha_2 C\alpha_4 S\alpha_6; & -C\alpha_2 S\alpha_3 S\alpha_5 - S\alpha_2 S\alpha_4 C\alpha_5; & -C\alpha_2 S\alpha_3 C\alpha_5 + S\alpha_2 S\alpha_4 S\alpha_5 \\ C\alpha_1 S\alpha_4 C\alpha_6 - S\alpha_1 S\alpha_3 S\alpha_6; & -C\alpha_1 S\alpha_4 S\alpha_6 - S\alpha_1 S\alpha_3 C\alpha_6; & C\alpha_1 C\alpha_4 C\alpha_5 - S\alpha_1 C\alpha_3 S\alpha_5; & -C\alpha_1 C\alpha_4 S\alpha_5 - S\alpha_1 C\alpha_3 C\alpha_5 \\ C\alpha_1 S\alpha_3 S\alpha_6 + S\alpha_1 S\alpha_4 C\alpha_6; & C\alpha_1 S\alpha_3 C\alpha_6 - S\alpha_1 S\alpha_4 S\alpha_6; & C\alpha_1 C\alpha_3 S\alpha_5 + S\alpha_1 C\alpha_4 C\alpha_5; & C\alpha_1 C\alpha_3 C\alpha_5 - S\alpha_1 C\alpha_4 S\alpha_5 \end{bmatrix}. \tag{2.12}$$

Here and in the following $C\alpha_1$, $S\alpha_1$, etc., stand for $\cos\alpha_1$, $\sin\alpha_1$ etc. By putting.

$$\left.\begin{aligned} \alpha_1 &= \tfrac{1}{2}(\varphi+\alpha), & \alpha_3 &= -\tfrac{1}{2}(\theta+\beta), & \alpha_5 &= \tfrac{1}{2}(\psi+\gamma), \\ \alpha_2 &= -\tfrac{1}{2}(\varphi-\alpha), & \alpha_4 &= -\tfrac{1}{2}(\theta-\beta), & \alpha_6 &= -\tfrac{1}{2}(\psi-\gamma). \end{aligned}\right\} \tag{2.13}$$

we get the following expression for L_{01} in terms of independent parameters on K and A:

$$\begin{aligned} L_{01} = {} & \frac{1}{S\theta}\left[C\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_1}{2} - S\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_2}{2}\right]\frac{\partial}{\partial\varphi} + \frac{1}{S\beta}\left[-S\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_1}{2} + C\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_2}{2}\right]\frac{\partial}{\partial\alpha} \\ & + \left[-C\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_1}{2} + S\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_2}{2}\right]\frac{\partial}{\partial\theta} + \left[-S\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_1}{2} + C\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_2}{2}\right]\frac{\partial}{\partial\beta} \\ & + \frac{1}{S\theta}\left[-C\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_1}{2} - S\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_2}{2}\right]\frac{\partial}{\partial\psi} + \frac{1}{S\beta}\left[S\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_1}{2} + C\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_2}{2}\right]\frac{\partial}{\partial\gamma} \\ & + \left[-S\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_1}{2} - C\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_2}{2}\right]\frac{\partial}{\partial\lambda}, \end{aligned} \tag{2.14}$$

where
$$\delta_1 = \gamma - \psi - \alpha + \varphi,$$
$$\delta_2 = \gamma - \psi + \alpha - \varphi.$$

The expressions for L_{02}, L_{03} and L_{04} are calculated similarly. One obtains

$$\begin{aligned} L_{02} = \frac{1}{S\theta}\left[C\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_1}{2}+S\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_2}{2}\right]\frac{\partial}{\partial\varphi}+\frac{1}{S\beta}\left[-S\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_1}{2}-C\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_2}{2}\right]\frac{\partial}{\partial\alpha} \\ +\left[C\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_1}{2}+S\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_2}{2}\right]\frac{\partial}{\partial\theta}+\left[S\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_1}{2}+C\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_2}{2}\right]\frac{\partial}{\partial\beta} \\ +\frac{1}{S\theta}\left[-C\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_1}{2}+S\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_2}{2}\right]\frac{\partial}{\partial\psi}+\frac{1}{S\beta}\left[S\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_1}{2}-C\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_2}{2}\right]\frac{\partial}{\partial\gamma} \\ +\left[S\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_1}{2}-C\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_2}{2}\right]\frac{\partial}{\partial\lambda}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} L_{03} = \frac{1}{S\theta}\left[-S\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_3}{2}-C\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_4}{2}\right]\frac{\partial}{\partial\varphi}+\frac{1}{S\beta}\left[-C\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_3}{2}-S\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_4}{2}\right]\frac{\partial}{\partial\alpha} \\ +\left[S\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_3}{2}+C\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_4}{2}\right]\frac{\partial}{\partial\theta}+\left[-C\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_3}{2}-S\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_4}{2}\right]\frac{\partial}{\partial\beta} \\ +\frac{1}{S\theta}\left[-S\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_3}{2}+C\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_4}{2}\right]\frac{\partial}{\partial\psi}+\frac{1}{S\beta}\left[C\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_3}{2}-S\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_4}{2}\right]\frac{\partial}{\partial\gamma} \\ +\left[-C\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_3}{2}+S\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_4}{2}\right]\frac{\partial}{\partial\lambda}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} L_{04} = \frac{1}{S\theta}\left[-S\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_3}{2}+C\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_4}{2}\right]\frac{\partial}{\partial\varphi}+\frac{1}{S\beta}\left[-C\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_3}{2}+S\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_4}{2}\right]\frac{\partial}{\partial\alpha} \\ +\left[-S\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_3}{2}+C\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_4}{2}\right]\frac{\partial}{\partial\theta}+\left[C\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_3}{2}-S\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_4}{2}\right]\frac{\partial}{\partial\beta} \\ +\frac{1}{S\theta}\left[-S\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_3}{2}-C\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_4}{2}\right]\frac{\partial}{\partial\psi}+\frac{1}{S\beta}\left[C\frac{\theta}{2}C\frac{\beta}{2}C\frac{\delta_3}{2}+S\frac{\theta}{2}S\frac{\beta}{2}C\frac{\delta_4}{2}\right]\frac{\partial}{\partial\gamma} \\ +\left[C\frac{\theta}{2}S\frac{\beta}{2}S\frac{\delta_3}{2}+S\frac{\theta}{2}C\frac{\beta}{2}S\frac{\delta_4}{2}\right]\frac{\partial}{\partial\lambda}, \end{aligned} \tag{2.17}$$

where
$$\delta_3=\gamma-\psi-\alpha-\varphi,$$
$$\delta_4=\gamma-\psi+\alpha+\varphi.$$

The operators of the compact subgroup are easily obtained in the form

$$\left.\begin{aligned} L_{12}&=T_3-S_3, & L_{13}&=S_2-T_2, & L_{14}&=S_1-T_1, \\ L_{34}&=T_3+S_3, & L_{24}&=S_2+T_2, & L_{23}&=-S_1-T_1, \end{aligned}\right\} \tag{2.18}$$

where
$$\left.\begin{aligned}
S_1 &= -C\varphi\frac{C\theta}{S\theta}\frac{\partial}{\partial\varphi}-S\varphi\frac{\partial}{\partial\theta}+\frac{C\varphi}{S\theta}\frac{\partial}{\partial\psi},\\
S_2 &= S\varphi\frac{C\theta}{S\theta}\frac{\partial}{\partial\varphi}-C\varphi\frac{\partial}{\partial\theta}-\frac{S\varphi}{S\theta}\frac{\partial}{\partial\psi},\\
S_3 &= -\frac{\partial}{\partial\varphi},\\
T_1 &= -C\alpha\frac{C\beta}{S\beta}\frac{\partial}{\delta\alpha}-S\alpha\frac{\partial}{\partial\beta}+\frac{C\alpha}{S\beta}\frac{\partial}{\partial\gamma},\\
T_2 &= S\alpha\frac{C\beta}{S\beta}\frac{\partial}{\partial\alpha}-C\alpha\frac{\partial}{\partial\beta}-\frac{S\alpha}{S\beta}\frac{\partial}{\partial\gamma},\\
T_3 &= -\frac{\partial}{\partial\alpha}.
\end{aligned}\right\} \tag{2.19}$$

The equations (2.14) – (2.19) thus give the desired explicit realization of the Lie algebra as differential operators on K and A. Note that the parameters λ and $(\psi+\gamma)$ do not appear in the coefficients of the differential operators.

3. A basis for an irreducible representation space

The number of labels which is required to characterize an arbitrary vector in a representation characterized by arbitrary values of the two invariants is six [1] whereas the number of parameters in the group spaces of K and A is seven. A definite choice of these parameters was made in the preceding section. The operators $\partial/\partial\lambda$ and $\partial/\partial\psi+\partial/\partial\gamma$ commute with all elements of $\mathfrak{l}(1,4)$ and we therefore consider functions for which these have constant values [1]. The eigenvalue of $\partial/\partial\lambda$ is denoted by $(ia-3/2)$, where a is an arbitrary complex number and we shall thus consider functions

$$e^{(ia-3/2)\lambda}f(\varphi,\theta,\psi,\alpha,\beta,\gamma)$$

which are eigenfunctions of $(\partial/\partial\psi+\partial/\partial\gamma)$. The corresponding eigenvalue is denoted by ib. Since $(\partial/\partial\psi+\partial/\partial\gamma)$ is connected with a periodic parameter in the group space of $S0(4)$ and we are going to consider one-valued and double-valued representations, it is sufficient to consider integer and half-integer values of b. At this stage it is convenient to make an explicit choice of a complete orthogonal system of functions in the group space of the covering group of $S0(4)$. The system we will consider is

$$\{e^{ih\varphi}P^j_{hk}(C\theta)\,e^{ik\psi}\,e^{im\alpha}\,P^l_{mk'}(C\beta)\,e^{ik'\gamma}\}, \tag{3.1}$$

where
$$j=0,\tfrac12,1,\tfrac32,\ldots\ |h|\leqslant j,\quad |k|\leqslant j,$$
$$l=0,\tfrac12,1,\tfrac32,\ldots\ |m|\leqslant l,\quad |k'|\leqslant l.$$

In (3.1) P^l_{mn} denotes the generalized spherical functions of $SO(3)$. The definition of these functions and a number of recursion relations for them, which will be needed later, are collected in an appendix. Since we shall use only functions for which $(\partial/\partial\psi+\partial/\partial\gamma)$ has the eigenvalue ib, we get $b=k+k'$. We are thus led to the following ansatz for the basis vectors in an irreducible representation space for $L(1,4)$:

$$|j,h,l,m\rangle = \sum_k C(j,l,k)\, e^{ih\varphi} P^j_{hk}\, e^{ik\psi}\, e^{im\alpha}\, P^l_{mb-k}\, e^{i(b-k)\gamma}, \tag{3.2}$$

where the sum goes over those k which fulfil

$$-j\leqslant k\leqslant j, \quad -l\leqslant b-k\leqslant l.$$

From (3.2) it is also obvious that if $b\geqslant 0$ we have $b\leqslant l+j$ and if $b<0$ we have $-b\leqslant l+j$. According to ref. [13] any unitary irreducible representation of $SO(4)$ can occur at most once in the decomposition of the unitary irreducible representations of $L(1,4)$ with respect to representations of $SO(4)$. If every separate term in the sum (3.2) could act as a $SO(4)$ basis from which bases for irreducible unitary representations of $L(1,4)$ could be built by taking a direct sum of such $SO(4)$ bases with increasing j and l, these $SO(4)$ bases would have a multiplicity equal to the number of allowed k values. We therefore expect that the irreducibility condition for the representations of $L(1,4)$ shall give conditions which connect $C(j,l,k)$'s with different k values, so that only one specific sum of the type (3.2) can be used to form a basis element for an irreducible representation of $L(1,4)$. Necessary conditions are then that I_2 and I_4 have the same values on all vectors $|j,h,l,\mathrm{m}\rangle$, independent of j,h,l and m. From the equation

$$I_2\,|j,h,l,m\rangle = \omega\,|j,h,l,m\rangle,$$

where
$$\begin{aligned} I_2 = \mathbf{T}^2+\mathbf{S}^2 &- \frac{\partial^2}{\partial\lambda^2} - 3\,\frac{\partial}{\partial\lambda} + 2\,\frac{C(\gamma-\psi)}{S\theta\, S\beta}\,\frac{\partial^2}{\partial\alpha\,\partial\varphi} \\ &+ 2\,\frac{S(\gamma-\psi)}{S\beta}\,\frac{\partial^2}{\partial\alpha\,\partial\theta} - 2\,\frac{C\theta C(\gamma-\psi)}{S\theta S\beta}\,\frac{\partial^2}{\partial\alpha\,\partial\psi} \\ &- 2\,\frac{S(\gamma-\psi)}{S\theta}\,\frac{\partial^2}{\partial\varphi\,\partial\beta} - 2\,\frac{C\beta C(\gamma-\psi)}{S\beta\, S\theta}\,\frac{\partial^2}{\partial\varphi\,\partial\gamma} \\ &+ 2\,C(\gamma-\psi)\,\frac{\partial^2}{\partial\beta\,\partial\theta} + 2\,\frac{C\theta}{S\theta}\,S(\gamma-\psi)\,\frac{\partial^2}{\partial\beta\,\partial\psi} \\ &- 2\,\frac{C\beta S(\gamma-\psi)}{S\beta}\,\frac{\partial^2}{\partial\theta\,\partial\gamma} + 2\left(1+\frac{C\theta\, C\beta}{S\theta\, S\beta}\,C(\gamma-\psi)\right)\frac{\partial^2}{\partial\gamma\,\partial\psi} \end{aligned}$$

if follows that $C(j,l,k)$ must satisfy the following recursion relation:

$$
\begin{aligned}
&(a^2+\tfrac{9}{4}-2k(b-k)-l(l+1)-j(j+1)-\omega)\,C(j,l,k)\\
&\quad=\sqrt{(j+k+1)(j-k)(l+b-k)(l-b+k+1)}\;C(j,l,k+1)\\
&\qquad+\sqrt{(j+k)(j-k+1)(l+b-k+1)(l-b+k)}\;C(j,l,k-1).
\end{aligned}\tag{3.3}
$$

Since a, b and ω are fixed in this relation, it can be used to determine ω in terms of a and b. For $b \geqslant 0$ we put $b = l + j$ and $k = j$ and get

$$\omega = a^2 + \tfrac{9}{4} - b^2 - b. \tag{3.4}$$

For $b < 0$ we put $b = -l - j$ and $k = -j$ and get

$$\omega = a^2 + \tfrac{9}{4} - b^2 + b, \tag{3.5}$$

i.e. ω depends only on $|b|$. Since, after the insertion of the value of ω, equation (3.3) contains both b and $|b|$, the recursion relations for $b \geqslant 0$ and $b < 0$ are different. However, the solution of both of these can be given in closed form. One finds for $b \geqslant 0$ and $b < 0$ respectively:

$$C(j,l,k) = i^{2k}\sqrt{\frac{(j+k)!\,(l+b-k)!}{(j-k)!\,(l-b+k)!}} \qquad b \geqslant 0, \tag{3.6}$$

$$C(j,l,k) = i^{2k}\sqrt{\frac{(j-k)!\,(l-b+k)!}{(j+k)!\,(l+b-k)!}} \qquad b < 0. \tag{3.7}$$

Of course, the $C(j,l,k)$'s are undetermined up to a factor depending only on j, l and b. Since the k-dependence of $C(j,l,k)$ is determined by (3.6) and (3.7), the condition

$$I_4\,|j,h,l,m\rangle = \varkappa\,|j,h,l,m\rangle$$

only gives $\varkappa$ in terms of a and b. In this way one finds

$$\varkappa = -(b^2 + |b|)(a^2 + \tfrac{1}{4}). \tag{3.8}$$

As an ansatz for a basis of a linear irreducible representation of $L(1,4)$ we can now use the set of vectors

$$\left.\begin{aligned} &\textstyle\sum \oplus\,|j,h,l,m\rangle \\ &j+l \geqslant |b|, \quad |h| \leqslant j, \quad |m| \leqslant l. \end{aligned}\right\} \tag{3.9}$$

The final test of the feasibility of this ansatz is now that the set (3.9) is invariant under the action of the operators of $\mathrm{l}(1,4)$. A straightforward calculation (for the necessary formulas concerning P^l_{mn}, see the appendix) shows that this is the case and the result is

$$
\begin{aligned}
S_3\,|j,h,l,m\rangle &= -ih\,|j,h,l,m\rangle \\
(iS_1+S_2)\,|j,h,l,m\rangle &= -\sqrt{(j-h)(j+h+1)}\,|j,h+1,l,m\rangle \\
(iS_1-S_2)\,|j,h,l,m\rangle &= -\sqrt{(j+h)(j-h+1)}\,|j,h-1,l,m\rangle \\
T_3\,|j,h,l,m\rangle &= -im\,|j,h,l,m\rangle \\
(iT_1+T_2)\,|j,h,l,m\rangle &= -\sqrt{(l-m)(l+m+1)}\,|j,h,l,m+1\rangle \\
(iT_1-T_2)\,|j,h,l,m\rangle &= -\sqrt{(l+m)(l-m+1)}\,|j,h,l,m-1\rangle,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&(iL_{02}+L_{01})\,|j,h,l,m\rangle \\
&\quad= i\,\frac{\sqrt{(j-h+1)(l+m+1)}}{(2j+1)(2l+1)}(j+l+1-|b|)(j+l-ia+\tfrac{3}{2})\,|j+\tfrac{1}{2},h-\tfrac{1}{2},l+\tfrac{1}{2},m+\tfrac{1}{2}\rangle \\
&\qquad+i\,\frac{\sqrt{(j-h+1)(l-m)}}{(2j+1)(2l+1)}(j-l-|b|)(j-l-ia+\tfrac{1}{2})\,|j+\tfrac{1}{2},h-\tfrac{1}{2},l-\tfrac{1}{2},m+\tfrac{1}{2}\rangle \\
&\qquad+i\,\frac{\sqrt{(j+h)(l+m+1)}}{(2j+1)(2l+1)}(j-l+|b|)(-j+l-ia+\tfrac{1}{2})\,|j-\tfrac{1}{2},h-\tfrac{1}{2},l+\tfrac{1}{2},m+\tfrac{1}{2}\rangle \\
&\qquad+i\,\frac{\sqrt{(j+h)(l-m)}}{(2j+1)(2l+1)}(j+l+1+|b|)(-j-l-ia-\tfrac{1}{2})\,|j-\tfrac{1}{2},h-\tfrac{1}{2},l-\tfrac{1}{2},m+\tfrac{1}{2}\rangle;
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&(iL_{02}-L_{01})\,|j,h,l,m\rangle \\
&\quad= i\,\frac{\sqrt{(j+h+1)(l-m+1)}}{(2j+1)(2l+1)}(j+l+1-|b|)(j+l-ia+\tfrac{3}{2})\,|j+\tfrac{1}{2},h+\tfrac{1}{2},l+\tfrac{1}{2},m-\tfrac{1}{2}\rangle \\
&\qquad+i\,\frac{\sqrt{(j+h+1)(l+m)}}{(2j+1)(2l+1)}(j-l-|b|)(j-l-ia+\tfrac{1}{2})\,|j+\tfrac{1}{2},h+\tfrac{1}{2},l-\tfrac{1}{2},m-\tfrac{1}{2}\rangle \\
&\qquad+i\,\frac{\sqrt{(j-h)(l-m+1)}}{(2j+1)(2l+1)}(j-l+|b|)(-j+l-ia+\tfrac{1}{2})\,|j-\tfrac{1}{2},h+\tfrac{1}{2},l+\tfrac{1}{2},m-\tfrac{1}{2}\rangle \\
&\qquad+i\,\frac{\sqrt{(j-h)(l+m)}}{(2j+1)(2l+1)}(j+l+1+|b|)(-j-l-ia-\tfrac{1}{2})\,|j-\tfrac{1}{2},h+\tfrac{1}{2},l-\tfrac{1}{2},m-\tfrac{1}{2}\rangle.
\end{aligned}
\tag{3.12}
$$

Similar expressions are valid for $(iL_{04}\pm L_{03})\,|j,h,l,m\rangle$. They are easily obtained from the commutation relations and the relations (3.10). In the coefficients of a given combination of $j\pm\frac{1}{2}$, $l\pm\frac{1}{2}$, only the square root factors are changed. Note that $(j+l)$ and $(j-l)$ are always changed by an integer. Equations (3.11) and (3.12) are valid both for $b\geqslant 0$ and $b<0$. Since only $|b|$ occurs in all rela-

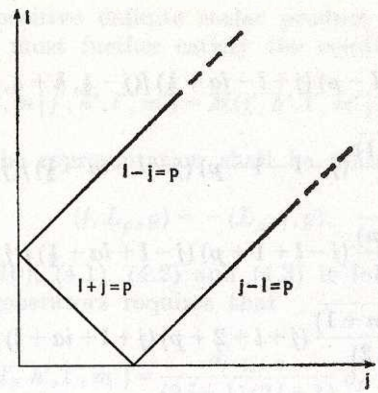

Fig. 1

tions determining the representations, $+b$ and $-b$ define the same representation. In the following we use the notation $|b|=p$. From (3.11) and (3.12) and the following remarks it is clear that the space formed by the vectors $|j,h,l,m\rangle$ with $j-l=-p, -p+1 \dots p-1, p$ and $j+l=p, p+1 \dots$ is invariant under $\mathrm{l}(1,4)$. Thus the irreducible representations of $SO(4)$ which are contained in an irreducible representation of $L(1,4)$ are, in the $j-l$-plane, found within or on the boundary of a semi-infinite strip according to Fig. 1.

In the next section it will be convenient to use not primarily the explicit expressions for the basis elements $|j,h,l,m\rangle$, but instead the sequence of coefficients which represent an arbitrary vector in the linear space built on

$$\{|j,h,l,m\rangle\}$$

$$j-l=-p,\ -p+1,\ \dots\ p-1,\ p,$$

$$j+l=\quad p,\ p+1,\ \dots,$$

$$h=-j,\ -j+1,\ \dots\ j-1,\ j,$$

$$m=-l,\ -l+1,\ \dots\ l-1,\ l.$$

We denote such a vector by f and introduce for the coefficients the notation $f(j,h,l,m)$. Thus

$$f=\sum f(j,h,l,m)\,|j,h,l,m\rangle.$$

Using (3.11) and (3.12), the coefficients $[(iL_{02}\pm L_{01})f](j,h,l,m)$ defined by

$$(iL_{02}\pm L_{01})f=\sum [(iL_{02}\pm L_{01})f](j,h,l,m)\,|j,h,l,m\rangle$$

can now be computed. The result is

$$
\begin{aligned}
&[(iL_{02}+L_{01})f](j,h,l,m)\\
&\quad = i\frac{\sqrt{(j-h)(l+m)}}{2j\cdot 2l}(j+l-p)(j+l-ia+\tfrac{1}{2})f(j-\tfrac{1}{2},h+\tfrac{1}{2},l-\tfrac{1}{2},m-\tfrac{1}{2})\\
&\qquad + i\frac{\sqrt{(j-h)(l-m+1)}}{2j(2l+2)}(j-l-1-p)(j-l-ia-\tfrac{1}{2})f(j-\tfrac{1}{2},h+\tfrac{1}{2},l+\tfrac{1}{2},m-\tfrac{1}{2})\\
&\qquad - i\frac{\sqrt{(j+h+1)(l+m)}}{(2j+2)2l}(j-l+1+p)(j-l+ia+\tfrac{1}{2})f(j+\tfrac{1}{2},h+\tfrac{1}{2},l-\tfrac{1}{2},m-\tfrac{1}{2})\\
&\qquad - i\frac{\sqrt{(j+h+1)(l-m+1)}}{(2j+2)(2l+2)}(j+l+2+p)(j+l+ia+\tfrac{3}{2})f(j+\tfrac{1}{2},h+\tfrac{1}{2},l+\tfrac{1}{2},m-\tfrac{1}{2});
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&[(iL_{02}-L_{01})f](j,h,l,m)\\
&\quad = i\frac{\sqrt{(j+h)(l-m)}}{2j\,2l}(j+l-p)(j+l-ia+\tfrac{1}{2})f(j-\tfrac{1}{2},h-\tfrac{1}{2},l-\tfrac{1}{2},m+\tfrac{1}{2})\\
&\qquad - i\frac{\sqrt{(j+h)(l+m+1)}}{2j(2l+2)}(j-l-1-p)(j-l-ia-\tfrac{1}{2})f(j-\tfrac{1}{2},h-\tfrac{1}{2},l+\tfrac{1}{2},m+\tfrac{1}{2})\\
&\qquad + i\frac{\sqrt{(j-h+1)(l-m)}}{(2j+2)2l}(j-l+1+p)(j-l+ia+\tfrac{1}{2})f(j+\tfrac{1}{2},h-\tfrac{1}{2},l-\tfrac{1}{2},m+\tfrac{1}{2})\\
&\qquad - i\frac{\sqrt{(j-h+1)(l+m+1)}}{(2j+2)(2l+2)}(j+l+2+p)(j+l+ia+\tfrac{3}{2})f(j+\tfrac{1}{2},h-\tfrac{1}{2},l+\tfrac{1}{2},m+\tfrac{1}{2}).
\end{aligned}
$$

The expressions for $[(iL_{04}\pm L_{03})f](j,h,l,m)$ will not be needed in the following and are therefore omitted.

4. The unitary irreducible representations of $L(1,4)$

In the preceding sections we have given a realization of the Lie algebra $\mathbf{l}(1,4$- either as a set of differential operators on the group spaces of K and A (sec) tion 2) or as a set of difference operators acting on the linear space spanned by the set $\{f(j,h,l,m)\}$ (section 3). It is this latter realization which will be used in this section. It should be noted that certain irreducibility conditions ($j+l\geqslant p$, $|j-l|\leqslant p$, $j+l$ integer or half-integer) are then automatically satisfied. The invariants take constant values determined by the parameters a and p. As we shall see there may however in certain cases be further limitations on the allowed values of the indices (j,l) due to the irreducibility condition.

Before discussing these questions it will be convenient to deal with the unitarity condition. Then we must, of course, introduce a scalar product into the linear space. Quite generally this can be done as follows

$$
(f,g)=\sum_{\substack{(jhlm)\\(j'h'l'm')}} \bar{f}(j,h,l,m)\,M(j,h,l,m\,|\,j',h',l',m')\,g(j',h',l',m'). \tag{4.1}
$$

In order to define a positive definite scalar product the matrix M has to be positive definite and it must further satisfy the condition

$$\bar{M}(j,h,l,m\,|\,j',h',l',m') = M(j',h',l',m'\,|\,j,h,l,m). \tag{4.2}$$

The requirement that the representation shall be unitary means that the generators $L_{\mu\nu}$ must satisfy

$$(f, L_{\mu\nu}\, g) = -(L_{\mu\nu} f, g). \tag{4.3}$$

From the equations (3.10), (4.1), (4.2) and (4.3) it follows that the anti-Hermiticity of the compact generators requires that

$$M(j,h,l,m\,|\,j',h',l',m') = \frac{d_{j+l,j-l}}{(2j+1)(2l+1)}\,\delta_{jj'}\,\delta_{ll'}\,\delta_{hh'}\,\delta_{mm'}, \tag{4.4}$$

where $d_{j+l,j-l}$ are real positive numbers. The unitarity condition

$$(L_{01}+iL_{02})^{\dagger} = -L_{01}+iL_{02}$$

then gives the following difference equations for $d_{JK}(J=j+l,\ K=j-l,\ a_1=\mathrm{Re}\,a,\ a_2=Im a)$

$$d_{J+1K}(J+1-p)(J+a_2+\tfrac{3}{2}) = d_{JK}(J+2+p)(J-a_2+\tfrac{3}{2}), \tag{4.5}$$

$$d_{J+1K}(J+1-p)\,a_1 = d_{JK}(J+2+p)\,a_1, \tag{4.6}$$

$$d_{JK+1}(K-p)(K+a_2+\tfrac{1}{2}) = d_{JK}(K+1+p)(K-a_2+\tfrac{1}{2}), \tag{4.7}$$

$$d_{JK+1}(K-p)\,a_1 = -d_{JK}(K+1+p)\,a_1. \tag{4.8}$$

In the discussion of these equations we can now limit ourselves to

$$J \geqslant p, \quad |K| \leqslant p \tag{4.9}$$

and treat separately integer and half-integer values of J (and K). Unitarity further requires a^2 to be real, i.e., a is real or imaginary.

1. *a real*

Assume first that $a_1 \neq 0$. Then the equations (4.5) and (4.6) imply that $a_2 = 0$ and that

$$\left.\begin{aligned} d_{J+1K} &= d_{JK}\,\frac{J+2+p}{J+1+p}, \\[2ex] d_{JK+1} &= d_{JK}\,\frac{K+1+p}{p-K} \end{aligned}\right\} \tag{4.10}$$

so that d_{JK} is everywhere positive for $J \geqslant p$ and $|K| \leqslant p$. Therefore we obtain in this case an irreducible unitary representation characterized by integer or half-

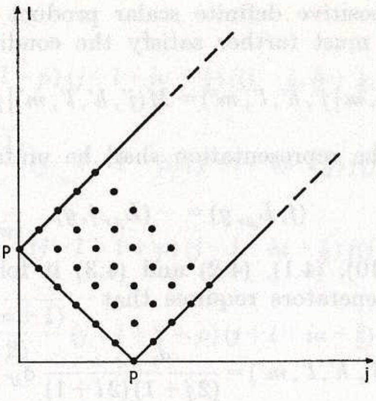

Fig. 2

integer p and $a_1 \neq 0$. The case $a_1 = 0$ and K integer can be adjoined trivially while the case $a_1 = 0$ and K half-integer is seen from equation (4.7) to give rise to two irreducible representations, one with $K \geqslant \frac{1}{2}$ and one with $K \leqslant -\frac{1}{2}$. These representations exhaust the possibilities when a is real. The (j, l) content of the representations corresponding to $a_1 \neq 0$ and $a_1 = 0$ and K integer is illustrated in Fig. 2.

2. *a imaginary*

The equations (4.6) and (4.8) are satisfied and from equation (4.7) it follows that $d_{JK} > 0$ for $J \geqslant p$, $|K| \leqslant p$ if K does not take the values $\pm\frac{1}{2}$ and if $|a_2| < \frac{1}{2}$. This means that one has an irreducible representation with the (j, l)-contents of Fig. 2 for p integer and $|a_2| < \frac{1}{2}$.

Further, if

$$a_2 = \overline{K} + \tfrac{1}{2}$$

then it is seen from equation (4.7) that K is bounded from above by $\overline{K} < 0$ and similarly when

$$a_2 = -\underline{K} + \tfrac{1}{2}$$

then K is bounded below by $\underline{K} > 0$. In this way we obtain irreducible representations with (j, l)-contents according to Figs. 3 and 4. Note that we here recover the cases $a = 0$ and K half-integer from 1.

There are two more special cases to be discussed. Both involve the restriction $K \equiv 0$. From equation (4.7) it follows that $d_{JO} > 0$ for $a_2 = \frac{1}{2}$ and p integer. The (j, l)-contents of this series of representations are illustrated in Fig. 5. However, if $p = 0$ then it *follows* from equation (4.9) that only $K = 0$ is allowed and equation (4.7) is irrelevant. Equation (4.5) then shows that the positive definiteness only requires $|a_2| < 3/2$. The (j, l)-contents of this series are given by Fig. 5 for $p = 0$.

In all our considerations on irreducibility we have used the fact that a vector is zero either if its components are zero or if the measure is zero (for those components which are not zero). With this in mind one can show that there

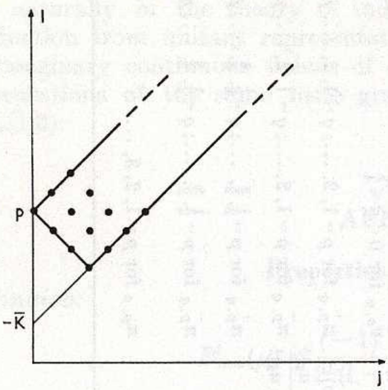

Fig. 3

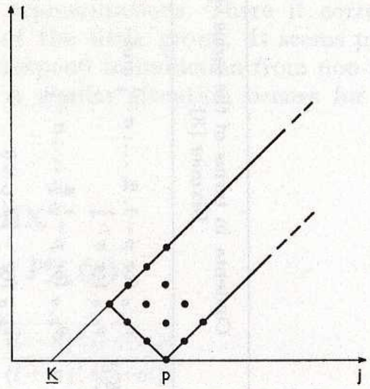

Fig. 4

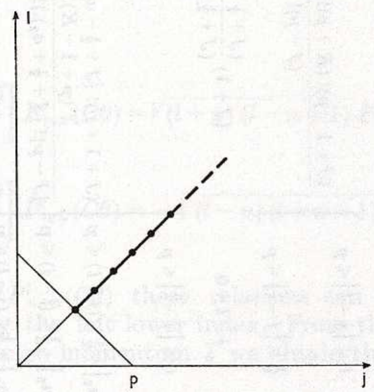

Fig. 5

exists to every discrete value of $a_2, a_2 = \overline{K} + \frac{1}{2}$ or $a_2 = -\underline{K} + \frac{1}{2}$, a positive value of the same absolute magnitude which corresponds to limits on the $f(j, h, l, m)$ instead of on d_{JK}. In the same way there corresponds to the value $a_2 = \frac{1}{2}$ connected with one of the last two special cases, a value $a_2 = -\frac{1}{2}$. By this observation we have restored a complete symmetry of the spectrum of a under the operation $a \to -a$. However, since the spectra of all operators of $\mathfrak{l}(1, 4)$ in a given representation depend only on $|a|$ and since the invariants involve only a^2, one might suspect that representations belonging to $\pm a$ are unitarily equivalent. A comparison with reference [8] shows that this is indeed the case.

The results are summarized in the table below, where we have also given the form of the kernel d_{JK} [cf. equations (4.5)–(4.8)]. The relation between our classification and that of [8] is also given. It is seen that all the unitary irreducible representations have actually been obtained by the present method. Finally we may mention that the series 1) and 2) correspond to what Takahashi [9] calls the first principal series and that the parameter $a_1(-\infty < a_1 < \infty)$ ap-

Single and double-valued representations of $L(1,4)$.

Notation of the series	Range of a_1 resp. a_2	Range of p	Range of $(j-l)$	Value of d_{JK}	Contents in terms of the series of Dixmier [8]
(1) $U_0(a_1, p)$	$\lvert a_1 \rvert > 0$	$p=0, 1, 2, \ldots$	$0 \leqslant \lvert j-l \rvert \leqslant p$	$\dfrac{(J+1+p)!\,(K+p)!\,(p+1-K)!}{(J-p)!}$	$\nu_{p,\sigma}$ for $p=1, 2, \ldots; \sigma > \frac{1}{4}$ and $\nu_{0,\sigma}$ for $\sigma > \frac{1}{4}$
(2) $U_{\frac{1}{2}}(a_1, p)$	$\lvert a_1 \rvert > 0$	$p=\frac{1}{2}, \frac{3}{2}, \ldots$	$\frac{1}{2} \leqslant \lvert j-l \rvert \leqslant p$		$\nu_{p,\sigma}$ for $p=\frac{1}{2}, \frac{3}{2}, \ldots; \sigma > \frac{1}{4}$
(3) $U_0(a_2, 0)$	$\frac{1}{2} \leqslant \lvert a_2 \rvert < \frac{3}{2}$	$p=0$	$j-l=0$	$(J+1)\,\dfrac{(J+\frac{1}{2}-a_2)!}{(J+\frac{1}{2}+a_2)!}$	$\nu_{0,\sigma}$ for $-2 < \sigma \leqslant 0$
(4) $U_0(a_2, p)$	$\lvert a_2 \rvert < \frac{1}{2}$	$p=0, 1, 2, \ldots$	$0 \leqslant \lvert j-l \rvert \leqslant p$		$\nu_{p,\sigma}$ for $p=1, 2, \ldots$, and $\nu_{0,\sigma}$ for $0 < \sigma \leqslant \frac{1}{4}$
(5) $U_0^+(a_2, p)$	$\lvert a_2 \rvert = \frac{1}{2}, \frac{3}{2}, \ldots, p-\frac{1}{2}$	$p=1, 2, \ldots$	$\lvert a_2 \rvert + \frac{1}{2} \leqslant (j-l) \leqslant p$	$\dfrac{(J+1+p)!\,(J+\frac{1}{2}-a_2)!\,(K+p)! \times (p+1-K)!\,(K-\frac{1}{2}-a_2)!}{(J-p)!\,(J+\frac{1}{2}+a_2).)!\,(K-\frac{1}{2}+a_2)!}$	$\pi^+_{p,q}$ for $p=1, 2, \ldots; q=1, 2, \ldots, p$
(6) $U_0^-(a_2, p)$	$\lvert a_2 \rvert = \frac{1}{2}, \frac{3}{2}, \ldots, p-\frac{1}{2}$	$p=1, 2, \ldots$	$\lvert a_2 \rvert + \frac{1}{2} \leqslant (l-j) \leqslant \mathrm{p}$		$\pi^-_{p,q}$ for $p=1, 2, \ldots; q=1, 2, \ldots, p$
(7) $U_{\frac{1}{2}}^+(a_2, p)$	$\lvert a_2 \rvert = 0, 1, \ldots, p-\frac{1}{2}$	$p=\frac{1}{2}, \frac{3}{2}, \ldots$	$\lvert a_2 \rvert + \frac{1}{2} \leqslant (j-l) \leqslant p$		$\pi^+_{p,q}$ for $p=\frac{1}{2}, \frac{3}{2}, \ldots; q=\frac{1}{2}, \frac{3}{2}, \ldots, p$
(8) $U_{\frac{1}{2}}^-(a_2, p)$	$\lvert a_2 \rvert = 0, 1, \ldots, p-\frac{1}{2}$	$p=\frac{1}{2}, \frac{3}{2}, \ldots$	$\lvert a_2 \rvert + \frac{1}{2} \leqslant (l-j) \leqslant p$		$\pi^-_{p,q}$ for $p=\frac{1}{2}, \frac{3}{2}, \ldots; q=\frac{1}{2}, \frac{3}{2}, \ldots, p$
(9) $U_0(\pm\frac{1}{2}, p)$	$\lvert a_2 \rvert = \frac{1}{2}$	$p=1, 2, \ldots$	$j-l=0$	$\dfrac{(J+1+p)!\,(J+\frac{1}{2}-a_2)!}{(J-p)!\,(J+\frac{1}{2}+a_2)!}$	$\pi_{p,0}$ for $p=1, 2, 3, \ldots$

pears naturally in the theory of induced representations. There it corresponds to induction from unitary representations of the little group. It seems probable that imaginary continuous values of a correspond to induction from non-unitary representations of the same little group. A similar situation occurs for L(1, 2) and L(1,3).

APPENDIX

Properties of the $P^l_{mn}(\mu)$

Definition:

$$P^l_{mn}(\mu) = \frac{(-1)^{l-n}}{2^l(l-n)!}\sqrt{\frac{(l-n)!\,(l+m)!}{(l+n)!\,(l-m)!}},$$

$$(1-\mu)^{-(m-n)/2}(1+\mu)^{-(m+n)/2}\frac{d^{l-m}}{d\mu^{l-m}}[(1-\mu)^{l-n}(1+\mu)^{l+n}].$$

Recursion relations:

$$\left[\frac{d}{d\theta}-\frac{m-nC\theta}{S\theta}\right]P^l_{mn}(C\theta)=\sqrt{(l+n)(l-n+1)}\,P^l_{mn-1}(C\theta),$$

$$\left[\frac{d}{d\theta}+\frac{m-nC\theta}{S\theta}\right]P^l_{mn}(C\theta)=-\sqrt{(l-n)(l+n+1)}\,P^l_{mn+1}(C\theta).$$

Using $P^l_{mn}(C\theta)=(-1)^{n-m}P^l_{n,m}(C\theta)$ these relations can be transformed into recursion relations involving the left lower index. From the addition of an angular momentum l and an angular momentum $\frac{1}{2}$ we obtain the following relations [14]

$$\begin{aligned}
C\theta/2\,P^l_{mn} &= C^{lm+\frac{1}{2}}_{11}\,C^{ln+\frac{1}{2}}_{11}\,P^{l+\frac{1}{2}}_{m+\frac{1}{2}n+\frac{1}{2}}+C^{lm+\frac{1}{2}}_{12}\,C^{ln+\frac{1}{2}}_{12}\,P^{l-\frac{1}{2}}_{m+\frac{1}{2}n+\frac{1}{2}},\\
C\theta/2\,P^l_{mn} &= C^{lm-\frac{1}{2}}_{21}\,C^{ln-\frac{1}{2}}_{21}\,P^{l+\frac{1}{2}}_{m-\frac{1}{2}n-\frac{1}{2}}+C^{lm-\frac{1}{2}}_{22}\,C^{ln-\frac{1}{2}}_{22}\,P^{l-\frac{1}{2}}_{m-\frac{1}{2}n-\frac{1}{2}},\\
S\theta/2\,P^l_{mn} &= C^{lm-\frac{1}{2}}_{21}\,C^{ln+\frac{1}{2}}_{11}\,P^{l+\frac{1}{2}}_{m-\frac{1}{2}n+\frac{1}{2}}+C^{lm-\frac{1}{2}}_{22}\,C^{ln+\frac{1}{2}}_{12}\,P^{l-\frac{1}{2}}_{m-\frac{1}{2}n+\frac{1}{2}},\\
S\theta/2\,P^l_{mn} &= -C^{lm+\frac{1}{2}}_{11}\,C^{ln-\frac{1}{2}}_{21}\,P^{l+\frac{1}{2}}_{m+\frac{1}{2}n-\frac{1}{2}}-C^{lm+\frac{1}{2}}_{12}\,C^{ln-\frac{1}{2}}_{22}\,P^{l-\frac{1}{2}}_{m+\frac{1}{2}n-\frac{1}{2}},
\end{aligned}$$

where C^{lm}_{ij} are the elements of the matrix

$$C^{lm}=\begin{bmatrix}\sqrt{\dfrac{l+m+\frac{1}{2}}{2l+1}} & -\sqrt{\dfrac{l-m+\frac{1}{2}}{2l+1}}\\[2ex] \sqrt{\dfrac{l-m+\frac{1}{2}}{2l+1}} & \sqrt{\dfrac{l+m+\frac{1}{2}}{2l+1}}\end{bmatrix}.$$

Note that it then follows that

$$C_{21}^{jk} C_{11}^{l\,b-k}\, C(j,l,k+\tfrac{1}{2}) - C_{11}^{jk} C_{21}^{l\,b-k}\, C(j,l,k-\tfrac{1}{2}) = i\,\frac{(j+l+1-p)}{\sqrt{(2j+1)(2l+1)}}\, C(j+\tfrac{1}{2},l+\tfrac{1}{2},k),$$

$$C_{21}^{jk} C_{12}^{l\,b-k}\, C(j,l,k+\tfrac{1}{2}) - C_{11}^{jk} C_{22}^{l\,b-k}\, C(j,l,k-\tfrac{1}{2}) = (-i)\,\frac{(j-l-p)}{\sqrt{(2j+1)(2l+1)}}\, C(j+\tfrac{1}{2},l-\tfrac{1}{2},k),$$

$$C_{22}^{jk} C_{11}^{l\,b-k}\, C(j,l,k+\tfrac{1}{2}) - C_{12}^{jk} C_{22}^{l\,b-k}\, C(j,l,k-\tfrac{1}{2}) = i\,\frac{(j-l+p)}{\sqrt{(2j+1)(2l+1)}}\, C(j-\tfrac{1}{2},l+\tfrac{1}{2},k),$$

$$C_{22}^{jk} C_{12}^{l\,b-k}\, C(j,l,k+\tfrac{1}{2}) - C_{12}^{jk} C_{22}^{l\,b-k}\, C(j,l,k-\tfrac{1}{2}) = (-i)\,\frac{(j+l+1+p)}{\sqrt{(2j+1)(2l+1)}}\, C(j-\tfrac{1}{2},l-\tfrac{1}{2},k),$$

where $C(j,l,k)$ stands for the expression given by (3.6) or (3.7).

ACKNOWLEDGEMENT

This work was completed while one of the authors (S. Ström) was a visitor at CERN. He wants to express his gratitude for the hospitality extended to him during this period. The financial support of M. E. Delbancos Teknologiska Fond is gratefully acknowledged.

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Tryckt den 1 april 1966

Uppsala 1966. Almqvist & Wiksells Boktryckeri AB

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STAFFAN STRÖM

Some remarks on the decomposition of a unitary representation of the Lorentz group with respect to representations of non-compact subgroups

ALMQVIST & WIKSELL

STOCKHOLM

1967

3.67 04 7 Communicated 22 February 1967 by L. Hulthén and N. Svartholm

Some remarks on the decomposition of a unitary representation of the Lorentz group with respect to representations of non-compact subgroups

By Staffan Ström

ABSTRACT

In the present article it is shown how infinitesimal methods can be used in decomposing a unitary irreducible representation of the Lorentz group, belonging to the main series, with respect to unitary representations of a non-compact subgroup. Two cases are considered, in which the non-compact subgroup is chosen as the three-dimensional Lorentz group and the two-dimensional Euclidean group respectively. The connection with the global approach is discussed.

1. Introduction

In the relativistic crossed channel partial wave analysis [1]–[5], where the total momentum is a spacelike vector, one encounters the group-theoretical problem of decomposing the unitary representations of the homogeneous Lorentz group $L(1,3)$ with respect to the unitary representation of the three-dimensional Lorentz group $L(1,2)$. In particular one is then interested in the limit of vanishing momentum [6], [7]. The decomposition process was studied by A. Sciarrino and M. Toller in a recent paper [8]. The present author has treated the same problem independently and arrived at the same results for the decomposition. However, we have used infinitesimal methods. We belive that this approach has an interest of its own which merits a presentation of it. The method exploited is the following. Starting from a realization of the Lie algebra $l(1,3)$ of $L(1,3)$ in terms of operators on a space of functions defined on the three-dimensional rotation group $R(3)$ we form the operator corresponding to the invariant of the non-compact subgroup in question (we treat also the case of the subgroup $E(2)$, the two-dimensional Euclidean group). The eigenfunctions of this operator are determined and finally the transformation from the eigenfunctions of the invariant of the compact subgroup $R(3)$ to the eigenfunctions of the invariant of the non-compact subgroup is constructed.

In section 2 we discuss some relevant operator realizations of $l(1,3)$. In section 3 we consider the decomposition of the unitary irreducible representations of $L(1,3)$ belonging to the main series with respect to unitary representations of $L(1,2)$. The connection with the global approach of reference [8] is discussed. Although there does not seem to exist any direct physical application of the representations of the Poincaré group corresponding to mass zero and the general (faithful) unitary irreducible representation of the little group $E(2)$, we give here in section 4 also the decomposition for this case.

2. On the global and infinitesimal form of the representations of $L(1, 3)$

In reference [9] two realizations of the unitary irreducible representations of $L(1, 3)$ are described. One of these gives explicitly the decomposition of the representation with respect to unitary irreducible representations of $R(3)$. The other is closely related to the decomposition with respect to the unitary representations of $E(2)$. In the present paper only those representations of $L(1, 3)$ which belong to the main series will be considered. They are characterized by two real numbers, l_0 and ν, where ν is arbitrary and $|l_0| = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The reason that we confine ourselves to a treatment of the main series is that the representations belonging to this series form a complete set [9]. The representations of $L(1, 3)$ will be described in terms of representations of $SL(2, C)$.

The first of the above-mentioned realizations uses the Iwasawa decomposition

$$a = u \cdot k, \tag{2.1}$$

which is valid for every element a of $SL(2, C)$. In equation (2.1) $u \in SU(2)$ and $k \in K$, the group of matrices of the form

$$k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix}, \quad \lambda, \mu \text{ complex numbers.} \tag{2.2}$$

We will use a convention slightly different from that used in [9] and consider left multiplication with the inverse element. In complete analogy with § 11 of [9] it then follows that a representation (l_0, ν) is given by the operator V_a, $a \in SL(2, C)$ defined by

$$V_a \varphi(u) = \frac{\alpha(a^{-1}u)}{\overline{\alpha(a^{-1}u)}} \varphi(a^{-1}u),$$

$$\alpha(a) = a_{11}^{-2l_0} \cdot |a_{11}|^{2(i\nu + l_0 - 1)}. \tag{2.3}$$

The product of two operators is defined as successive multiplication of $u \in SU(2)$ from the left with the inverse element. The reader is referred to [9] for more details concerning the notation, the definition of the scalar product, the construction of a set of basis functions etc.

The second realization uses the decomposition

$$a = z \cdot k \tag{2.4}$$

where $k \in K$ and $z \in Z$, the group of matrices of the form

$$z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}. \tag{2.5}$$

The decomposition (2.4) is valid for all elements a with $a_{11} \neq 0$. The invariant measure of the elements with $a_{11}=0$ is zero and therefore it will be sufficient to consider elements a for which (2.4) is valid. Again, in complete analogy with § 10 of [9], a representation (l_0, ν) is given by the operator V_a defined by

$$V_a f(z) = (-a_{12}z + a_{22})^{-2l_0} \cdot |-a_{12}z + a_{22}|^{2(i\nu + l_0 - 1)} \cdot f\left(\frac{a_{11}z - a_{21}}{-a_{12}z + a_{22}}\right). \tag{2.6}$$

Note that the group of matrices of the form

$$\begin{pmatrix} e^{-i(\psi/2)} & 0 \\ z & e^{i(\psi/2)} \end{pmatrix} \tag{2.7}$$

is the "spinor group" associated with an $E(2)$-subgroup of $L(1,3)$.

From (2.3) it is of course easy to derive a realization of the generators of $L(1,3)$ in termt of differential operators and multipliers expressed in the variables of $SU(2)$ (or $R(3)$). However, such a realization can also be obtained without any knowledge of the global form (2, 3). In particular, it may be obtained from the representation of $L(1,3)$ as a transformation group on its parameters

$$g \xrightarrow{g_1} (g_1)^{-1} g \tag{2.8}$$

if one uses the Iwasawa decomposition (2.1) for the elements and then makes use of the possibility of omitting the terms involving differential operators with respect to the parameters of the nilpotent subgroup [10], [11]. Another parametrization of $L(1,3)$ which is often useful is the following

$$g = r_1 \cdot a \cdot r_2, \tag{2.9}$$

where $r_1, r_2 \in R(3)$ and a is an acceleration in a prescribed direction, e.g. the z-direction. In [12], [13] we have studied the matrix elements in a representation (l_0, ν) of the operator V_g with g parametrized according to (2.9). The parameters may be chosen as follows

$$g = r_z(\varphi)\, r_y(\theta)\, r_z(\psi)\, a_z(\alpha)\, r_y(\beta)\, r_z(\gamma). \tag{2.10}$$

From (2.8) and (2.10) we obtain the following realization of the elements $\mathbf{M}$ and $\mathbf{N}$ of $l(1,3)$:

$$M_1 = i\left(\frac{c\varphi c\theta}{s\theta}\,\partial_\varphi + s\varphi\partial_\theta - \frac{c\varphi}{s\theta}\,\partial_\psi\right),$$

$$M_2 = i\left(\frac{s\varphi c\theta}{s\theta}\,\partial_\varphi - c\varphi\partial_\theta - \frac{s\varphi}{s\theta}\,\partial_\psi\right),$$

$$M_3 = -i\partial_\varphi,$$

$$N_1 = i\left(\frac{s\varphi \operatorname{Ch}\alpha}{s\theta \operatorname{Sh}\alpha}\partial_\varphi - \frac{c\varphi c\theta \operatorname{Ch}\alpha}{\operatorname{Sh}\alpha}\partial_\theta - c\varphi s\theta\,\partial_\alpha\right.$$
$$-\left[\frac{c\beta}{s\beta \operatorname{Sh}\alpha}(s\varphi c\psi + c\varphi s\psi c\theta) + \frac{s\varphi c\theta \operatorname{Ch}\alpha}{s\theta \operatorname{Sh}\alpha}\right]\partial_\psi$$
$$\left. - \frac{s\varphi s\psi - c\varphi c\psi c\theta}{\operatorname{Sh}\alpha}\partial_\beta + \frac{s\varphi c\psi + c\varphi s\psi c\theta}{s\beta \operatorname{Sh}\alpha}\partial_\gamma\right),$$

$$N_2 = i\left(-\frac{c\varphi \operatorname{Ch}\alpha}{s\theta \operatorname{Sh}\alpha}\partial_\varphi - \frac{s\varphi c\theta \operatorname{Ch}\alpha}{\operatorname{Sh}\alpha}\partial_\theta - s\varphi s\theta\,\partial_\alpha\right.$$
$$+\left[\frac{c\beta}{s\beta \operatorname{Sh}\alpha}(c\varphi c\psi - s\varphi s\psi c\theta) + \frac{c\varphi c\theta \operatorname{Ch}\alpha}{s\theta \operatorname{Sh}\alpha}\right]\partial_\psi$$
$$\left. + \frac{c\varphi s\psi + s\varphi c\psi c\theta}{\operatorname{Sh}\alpha}\partial_\beta - \frac{c\varphi c\psi - s\varphi s\psi c\theta}{s\beta \operatorname{Sh}\alpha}\partial_\gamma\right),$$

$$N_3 = i\left(\frac{s\theta \operatorname{Ch}\alpha}{\operatorname{Sh}\alpha}\partial_\theta - c\theta\,\partial_\alpha + \frac{s\theta s\psi c\beta}{s\beta \operatorname{Sh}\alpha}\partial_\psi - \frac{s\theta c\psi}{\operatorname{Sh}\alpha}\partial_\beta - \frac{s\theta s\psi}{s\beta \operatorname{Sh}\alpha}\partial_\gamma\right), \tag{2.11}$$

where $s\varphi \equiv \sin\varphi$, $c\varphi \equiv \cos\varphi$, $\partial_\varphi \equiv \partial/\partial_\varphi$ etc. We note that by passing formally to the limit $\alpha \to \infty$ the parameters ψ, α, β and γ disappears from the coefficients in (2.11) and all terms containing ∂_β and ∂_γ disappear. In the remaining expression ∂_ψ and ∂_α can thus be put equal to constants. Using the discussion in [12] concerning the asymptotic behaviour of the matrix elements and the use of these elements in constructing a basis in a representation space, it follows that ∂_α shall be replaced by $(i\nu - 1)$ and ∂_ψ by il_0. In a representation (l_0, ν) of $L(1,3)$ the values of the invariants are

$$\begin{aligned} \mathbf{M}^2 - \mathbf{N}^2 &= l_0^2 - 1 - \nu^2, \\ \mathbf{M}\cdot\mathbf{N} &= l_0 \cdot \nu. \end{aligned} \tag{2.12}$$

This is obtained from the structure of $l(1,3)$ and from the hermiticity condition (cf. [9]). It is easy to check that the realization of $\mathbf{M}$ and $\mathbf{N}$ constructed from (2.11) according to the above mentioned rules satisfies (2.12) i.e. it corresponds to a representation (l_0, ν). We see here how the multiplier representation can be constructed as a certain limiting case of the realization which uses a set of matrix elements as basis functions.

Thus the following realization of the generators $\mathbf{M}$ and $\mathbf{N}$ of a representation (l_0, ν) of $L(1,3)$ will be used:

$$M_1 = i\left(\frac{c\varphi c\theta}{s\theta}\partial_\varphi + s\varphi\,\partial_\theta - il_0\frac{c\varphi}{s\theta}\right),$$

$$M_2 = i\left(\frac{s\varphi c\theta}{s\theta}\partial_\varphi - c\varphi\,\partial_\theta - il_0\frac{s\varphi}{s\theta}\right),$$

$$M_3 = -i\,\partial_\varphi,$$

$$N_1 = i\left(\frac{s\varphi}{s\theta}\partial_\varphi - c\varphi c\theta\,\partial_\theta - il_0\frac{s\varphi c\theta}{s\theta} - (i\nu-1)\,c\varphi s\theta\right),$$

$$N_2 = i\left(-\frac{c\varphi}{s\theta}\partial_\varphi - s\varphi c\theta\,\partial_\theta + il_0\frac{c\varphi c\theta}{s\theta} - (i\nu-1)\,s\varphi s\theta\right),$$

$$N_3 = i\,(s\theta\,\partial_\theta - (i\nu-1)\,c\theta). \tag{2.13}$$

The representation space of a representation (l_0, ν) can be constructed as a direct sum

$$\sum_{l=|l_0|,\, m\leqslant l}^{\infty} \oplus\, |l, m; l_0, \nu\rangle$$

of finite-dimensional spaces

$$|l, m; l_0, \nu\rangle = N(l, l_0, \nu)\, D^l_{m,l_0}(\varphi, \theta, \psi),$$

Here D^l_{m,l_0} are the rotation matrices i.e. the representation space has as basis elements eigenfunctions of $\mathbf{M}$ and M_3. In connection with the reduction problem we will consider eigenfunctions of the invariants of the $L(1,2)$-group generated by M_3, N_1 and N_2 and of the $E(2)$-group generated by M_3, $P_1 = M_1 - N_2$ and $P_2 = M_2 + N_1$. This is the $E(2)$-group which corresponds to the subgroup (7) of $SL(2, C)$. In the realization (2.13) the invariants are represented by the following operators:

$$Q = N_1^2 + N_2^2 - M_3^2 = (-1)\left[c^2\theta\,\partial_\theta^2 + \left(\frac{c\theta}{s\theta} + (2(i\nu-1)-1)\,c\theta s\theta\right)\partial_\theta \right.$$
$$\left. + s^2\theta(i\nu-1)^2 + (1+c^2\theta)(i\nu-1) + \frac{c^2\theta}{s^2\theta}\partial_\varphi^2 - \frac{c^2\theta}{s^2\theta}l_0^2 - 2il_0\partial_\varphi\right], \tag{2.14}$$

$$P_1^2 + P_2^2 = (-1)\left[(1+c\theta)^2\partial_\theta^2 + \left\{2\left(\frac{c\theta}{s\theta} + (i\nu-1)\,s\theta\right)(1+c\theta) - c\theta s\theta\right\}\partial_\theta \right.$$
$$\left. + (i\nu-1)^2\cdot s^2\theta + (i\nu-1)(1+c\theta)^2 - \left(\frac{1+c\theta}{s\theta}\right)^2(l_0^2 - \partial_\varphi^2 + 2il_0\partial_\varphi)\right]. \tag{2.15}$$

3. The decomposition of a representation (l_0, ν) of $L(1,3)$ with respect to unitary representations of $L(1,2)$

Let q denote the eigenvalue of Q in any one of the unitary irreducible representations of $L(1,2)$ which occur in the completeness relation (the representation theory of $L(1,2)$ will not be reviewed here, the reader is referred to [14], [15], [16]). In the decomposition problem the multiplier realization of $\mathbf{M}$ and $\mathbf{N}$ discussed in section 2 and given explicitly by equation (2.13) will be used, i.e. we determine the eigenfunctions of Q and M_3 where Q is given by (2.14). Thus the eigenfunctions can be written as

$$F(\varphi, \theta, \psi; q, m; l_0, \nu) = e^{im\varphi} F^q_{m,l_0}(\theta; \nu)\, e^{il_0\varphi}$$

and the eigenvalue equation

$$QF(\varphi, \theta, \psi; q, m; l_0, \nu) = qF(\varphi, \theta, \psi; q, m; l_0, \nu)$$

now reads

$$\Big[c^2\theta\,\partial_\theta^2 + \Big(\frac{c\theta}{s\theta} + (2(i\nu-1)-1)\,c\theta\,s\theta\Big)\,\partial_\theta + s^2\theta(i\nu-1)^2 + (1+c^2\theta)\,(i\nu-1)$$
$$-(m^2+l_0^2)\,\frac{c^2\theta}{s^2\theta} + 2ml_0\,\frac{c\theta}{s^2\theta} + q\Big]\,F^q_{m,l_0}(\theta;\nu) = 0. \quad (3.1)$$

Here it is understood that for the discrete series of representations m is appropriately bounded by q. An analysis of equation (3.1) shows that it is advantageous to first extract a factor $|c\theta|^{i\nu-1}$. With $F^q_{m,l_0}(\theta;\nu) = |c\theta|^{i\nu-1}\cdot G^q_{m,l_0}(\theta;\nu)$ it follows for $\theta \neq \pi/2$ that $G^q_{m,l_0}(\theta,\nu)$ satisfies

$$\Big[c^2\theta\,\partial_\theta^2 + \Big(\frac{c\theta}{s\theta} - c\theta\,s\theta\Big)\,\partial_\theta + 2ml_0\,\frac{c\theta}{s^2\theta} - (m^2+l_0^2)\,\frac{c^2\theta}{s^2\theta} + q\Big]\,G^q_{m,l_0}(\theta;\nu) = 0. \quad (3.2)$$

We now distinguish between the two cases:

$$\text{i)}\quad 0 < \theta < \tfrac{1}{2}\pi, \qquad \text{and} \qquad \text{ii)}\quad \tfrac{1}{2}\pi < \theta < \pi.$$

Consider first item i). With $s_1 = c\theta^{-1}$ and $G^q_{m,l_0}(\theta;\nu) = H^{1,q}_{m,l_0}(s_1;\nu)$ equation (3.2) reads

$$\Big[(s_1^2-1)\,\frac{d}{ds_1^2} + 2s_1\,\frac{d}{ds_1} + \frac{2ml_0 s_1}{s_1^2-1} - \frac{m^2+l_0^2}{s_1^2-1} + q\Big]\,H^{1,q}_{m,l_0}(s_1;\nu) = 0. \quad (3.3)$$

This is equation which is obtained for the matrix elements of an acceleration with velocity $v = \operatorname{Tgh}\alpha$, where $s_1 = \operatorname{Ch}\alpha$, in a representation (q) of $L(1,2)$ in a basis where M_3 is diagonal [14] (we will return later to the fact that it is actually an acceleration along the x-axis). Note that it follows that for the discrete series of representations also l_0 is bounded by q in the same way as m. Since l_0 is given this results in a restriction on the allowed q-values for the discret series. By considering all the allowed q-values and the corresponding allowed m-values we get a complete set of functions over the carrier space (φ, α), subject to the condition that l_0 is fixed.

Consider now item ii). In this case we put $s_2 = -(c\theta)^{-1}$ and get for $G^q_{m,l_0}(\theta;\nu) = H^{2,q}_{m,l_0}(s_2,\nu)$ an equation which is identical with the one for $H^{1,q}_{m,-l_0}(s_2;\nu)$. Consequently the only difference between i) and ii) is a change of sign in l_0 and we get in the same way a complete set of functions.

Thus we have seen that in each of the intervals $0 < \theta < \frac{1}{2}\pi$ and $\frac{1}{2}\pi < \theta < \pi$ we can construct the eigenfunctions of the invariant of the non-compact subgroup $L(1,2)$. The essential connection is given by the change of variable $|c\theta| = (\operatorname{Ch}\alpha)^{-1}$. Here θ denotes the angle of rotation around the y axis (cf. (2.10)) whereas α can be regarded as describing an acceleration in the x-directions with velocity $v = \operatorname{Tgh}\alpha$ (see below). Points with $\theta = \frac{1}{2}\pi$ have zero invariant measure in the carrier space (φ, θ) and therefore they need not be considered. As expected the functions

$F^q_{m,l_0}(\theta;\nu)$ are not square integrable over the intervals $(0, \frac{1}{2}\pi)$ and $(\frac{1}{2}\pi, \pi)$ respectively with the measure $s\theta d\theta$, when q denotes a representation belonging to the continuous class (cf. [14]). From the division of the interval $[0, \pi]$ of θ into two subintervals, $[0, \frac{1}{2}\pi]$, $[\frac{1}{2}\pi, \pi]$ in each of which we have defined a variable α connected with the $L(1,2)$ subgroups it follows that the representation space of a representation (l_0, ν) is split into the direct sum of two spaces, each of which is an integral and sum of unitary irreducible representations of $L(1,2)$.

As we have remarked above a realization of a basis for a representation space of (l_0, ν) is

$$|lm; l_0, \nu\rangle = \mathrm{N}(l, l_0, \nu)\, e^{im\varphi}\, P^l_{m,l_0}(c\theta)\, e^{il_0\psi}.$$

The integral transform which transforms these eigenfunctions of $\mathbf{M}^2$ and M_3 into the eigenfunctions $F(\varphi, \theta, \psi; q, m; l_0, \nu)$ of Q and M_3 will effectively be a transform from $P^l_{m,l_0}(c\theta)$ to $F^q_{m,l_0}(\theta;\nu)$. This transform will involve a summation and integration over the representations (q). The notation $\int dq$ will be used for this operation (for more details concerning these facts, see [15], [16] and also [8]; we will need only the simplest formal properties of the completeness relation for the representations (q)). Thus we shall consider the expansions

i) $$0 \leqslant \theta < \tfrac{1}{2}\pi, \quad c\theta = (\mathrm{Ch}\,\alpha)^{-1},$$

$$P^l_{m,l_0}(c\theta) = (\mathrm{Ch}\,\alpha)^{-(i\nu-1)} \cdot \int dq\, C_2(q; l, m; l_0, \nu)\, H^{1,q}_{m,l_0}(\mathrm{Ch}\,\alpha; \nu). \tag{3.4 a}$$

ii) $$\tfrac{1}{2}\pi < \theta \leqslant \pi, \quad c\theta = -(\mathrm{Ch}\,\alpha)^{-1},$$

$$P^l_{m,l_0}(c\theta) = (\mathrm{Ch}\,\alpha)^{-(i\nu-1)} \cdot \int dq\, C_2(q; l, m; l_0, \nu)\, H^{2,q}_{m,l_0}(\mathrm{Ch}\,\alpha; \nu). \tag{3.4 b}$$

Using the completeness relation and the orthogonality properties of the matrix elements $H^{1,q}_{m,l_0}$ and $H^{2,q}_{m,l_0}$ respectively (cf. [14]) the equations (3.4 a) and (3.4 b) can be inverted to give

$$C_1(q; l, m; l_0, \nu) = \int_0^\infty (\mathrm{Ch}\,\alpha)^{i\nu-1} \cdot \overline{H^{1,q}_{m,l_0}(\mathrm{Ch}\,\alpha;\nu)} \cdot P^l_{m,l_0}((\mathrm{Ch}\,\alpha)^{-1})\, \mathrm{Sh}\,\alpha\, d\alpha, \tag{3.5 a}$$

$$C_2(q; l, m; l_0, \nu) = \int_0^\infty (\mathrm{Ch}\,\alpha)^{i\nu-1} \cdot \overline{H^{1,q}_{m,-l_0}(\mathrm{Ch}\,\alpha;\nu)} \cdot P^l_{m,l_0}(-(\mathrm{Ch}\,\alpha)^{-1})\, \mathrm{Sh}\,\alpha\, d\alpha. \tag{3.5 b}$$

With the conventions used here the explicit expression for $P^l_{m,l_0}(c\theta)$ is

$$P^l_{m,l_0}(c\theta) = \frac{(-1)^{l-l_0}}{2^l (l-l_0)!}\left[\frac{(l-l_0)!\,(l+m)!}{(l+l_0)!\,(l-m)!}\right] \cdot (1-c\theta)^{-((m-l_0)/2)} \cdot (1+c\theta)^{-((m+l_0)/2)}$$
$$\times \frac{d^{l-m}}{d(c\theta)^{l-m}}\left[(1-c\theta)^{l-l_0}(1+c\theta)^{l+l_0}\right], \tag{3.6}$$

i.e. $$P^l_{m,l_0}(-c\theta) = (-1)^{l+m} P^l_{m,-l_0}(c\theta)$$

and thus $$C_2(q; lm; l_0, \nu) = (-1)^{l+m} C_1(q; lm; -l_0, \nu).$$

Summarizing we may say that by solving equation (3.3) and determining the transformation functions C_1 and C_2 we have derived the essential features of the decomposition and we see that the result agrees with that of reference [8] (apart from some differences in conventions). As is shown in [8] C_1 and C_2 may be expressed in terms of generalized hypergeometric functions. Since the properties of C_1 and C_2 are treated thoroughly and extensively in [8] there is no need for us to make further comments on them here.

The present infinitesimal method has also been applied to the problem of decomposing a representation (q) with respect to the unitary irreducible representations of $L(1,1)$, the two-dimensional Lorentz group. The results are reported elsewhere [17].

We would like to add a few more remarks on the connection between the present approach and that used in reference [8]. If the variable $\operatorname{Ch}\alpha = (c\theta)^{-1}$, $0 < \theta < \frac{1}{2}\pi$, is introduced in the expressions for M_3, N_1 and N_2 in (2.13) a realization of these as operators on functions of φ and α is obtained. If furthermore a factor $(\operatorname{Ch}\alpha)^{-(i\nu-1)}$, corresponding to the multiplier in the representation (l_0, ν) is extracted from the function space considered and $\partial\psi = il_0$ is re-introduced the following realization of $l(1,2)$, the Lie algebra of $L(1,2)$, is obtained:

$$M_3 = -i\partial_\varphi,$$

$$N_1 = i\left(s\varphi\frac{\operatorname{Ch}\alpha}{\operatorname{Sh}\alpha}\partial_\varphi - c\varphi\partial_\alpha - \frac{s\varphi}{\operatorname{Sh}\alpha}\partial_\psi\right),$$

$$N_2 = i\left(-c\varphi\frac{\operatorname{Ch}\alpha}{\operatorname{Sh}\alpha}\partial_\varphi - s\varphi\partial_\alpha + \frac{c\varphi}{\operatorname{Sh}\alpha}\partial_\psi\right). \tag{3.7}$$

This is, as is easily checked, the realization of $l(1,2)$ which one obtains from the left regular representation of $L(1,2)$ defined on functions of an element g of $L(1,2)$ parametrized as follows

$$g = r_z(\varphi)\, a_x(\alpha)\, r_z(\psi).$$

Analogous results are obtained for the interval $\frac{1}{2}\pi < \theta < \pi$, the only difference being a change of sign of ψ. In each of the intervals $0 \leqslant \theta < \frac{1}{2}\pi$ and $\frac{1}{2}\pi < \theta \leqslant \pi$ we can thus with every element of $R(3)$ associate an element of $L(1,2)$. These observations should be compared with the discussion in reference [8] of the double cosets of $SL(2,C)$ with respect to the subgroups K and $SU(1,1)$.

In conclusion we remark that one could also consider another approach which is in a sense the inverse of the one used above. The procedure we refer to is the one in which one starts with a realization of $l(1,3)$ in multipler form which contains a realization of the type (3.7) for a $l(1,2)$ subalgebra and which contains multiplier terms in N_3, M_1 and M_2. One can then determine the eigenfunctions of $\mathbf{M}^2$ as functions of φ, α and ψ. The difficulty with this approach is that one has then a realization of $l(1,3)$ which can not be integrated to a representation of $L(1,3)$. This is of course just another aspect of that the elements of the form $v \cdot k$, $v \in SU(1,1)$, $k \in K$ do not give all elements of $SL(2,C)$ but only elements in a neighbourhood of the identity. This leads to the necessity of doubling the function space contidered in order to achieve ortogonality and completeness (as is implied by the results of this section).

4. The decomposition of a representation (l_0, ν) of $L(1,3)$ with respect to unitary representations of $E(2)$

The problem will be treated with the infinitesimal method used in the previous section but it will also be indicated how the global approach can be applied in this case. Readers not familiar with the representation theory of $E(2)$ are referred to [18] for a review.

According to (2.13) and (2.14) we shall consider eigenfunctions $\exp(im\varphi) \times T^p_{m,l_0}(\theta;\nu)\exp(il_0\psi)$ of $P_1^2+P_2^2$ and M_3 with eigenvalues p^2 and m respectively where $T^p_{m,l_0}(\theta;\nu)$ satisfies

$$\Bigg[(1+c\theta)^2\partial_\theta^2+\left\{2\left(\frac{c\theta}{s\theta}+(i\nu-1)\,s\theta\right)(1+c\theta)-c\theta\, s\theta\right\}\partial_\theta + (i\nu-1)^2 s^2\theta+(i\nu-1)(1+c\theta)^2-\left(\frac{1+c\theta}{s\theta}\right)^2\cdot(m-l_0)^2+p^2\Bigg]\,T^p_{m,l_0}(\theta;\nu)=0. \quad (4.1)$$

With $T^p_{m,l_0}(\theta;\nu)=(1+(\theta)^{i\nu-1}\cdot L^p_{m,l_0}(\theta;\nu)$ it follows that $L^p_{m,l_0}(\theta;\nu)$ satisfies

$$\left[\partial_\theta^2+\frac{c\theta}{s\theta}\partial_\theta-\left(\frac{m-l_0}{s\theta}\right)^2+\frac{p^2}{(1+c\theta)^2}\right]L^p_{m,l_0}(\theta;\nu=0. \quad (4.2)$$

In the variable $a=\operatorname{tg}\theta/2$ and with $L^p_{m,l_0}(\theta;\nu)=K^p_{m,l_0}(a;\nu)$ (4.2) reads

$$\left[\partial_a^2+\frac{1}{a}\partial_a-\left(\frac{m-l_0}{a}\right)^2+p^2\right]K^p_{m,l_0}(a;\nu)=0,$$

i.e. the solutions are Bessel functions $J_{m-l_0}(ap)$. An arbitrary element g of $E(2)$ can be written as

$$g=r_z(\varphi)\,t_{y'}(a)\,r_z(\psi),$$

where $t_{y'}(a)$ is a translation of length a in the y'-direction. $E(2)$ is then supposed to be a transformation group in a $x'-y'$-plane, perpendicular to the z-axis. The matrix elements of a translation $t_{y'}(a)$ in a representation characterized by $P_1^2+P_2^2=p^2$, computed in a basis where M_3 is diagonal is just the Bessel functions $J_{m-l_0}(ap)$ and these matrix elements are orthogonal:

$$\int_0^\infty J_{m-l_0}(ap')\,J_{m-l_0}(ap)\,a\,da=\frac{1}{p}\,\delta(p-p')$$

and form a complete set:

$$\int_0^\infty J_{m-l_0}(a'p)\, J_{m-l_0}(ap)\, p\, dp = \frac{1}{a}\,\delta(a-a').$$

The integral transformation which expresses $P^l_{m,l_0}(c\theta)$ in terms of the eigenfunctions of $P_1^2+P_2^2$ reads

$$P^l_{m,l_0}(c\theta) = (1+c\theta)^{i\nu-1}\int_0^\infty p\, dp\, t(p; l, m; l_0, \nu)\, J_{m-l_0}(p\cdot \operatorname{tg}\theta/2)$$

from which one easily deduces

$$t(p; l, m; l_0, \nu) = \int_0^\infty a\, da \left(\frac{1+a^2}{2}\right)^{i\nu-1} P^l_{m,l_0}\left(\frac{1-a^2}{1+a^2}\right) J_{m-l_0}(ap). \tag{4.3}$$

From (4.3) and formulas (5, 6, 3) of [19] and (20, 5, 4) of [20] it follows that $t(p; l, m; l_0, \nu)$ can be expressed as a sum of Meijer's G-functions in the following way:

$$t(p; l, m; l_0, \nu) = (-1)^{l-l_0}\, 2^{-i\nu}\left(\frac{p}{2}\right)^{-m-l_0} \frac{[(l-l_0)!\,(l+l_0)!\,(l-m)!\,(l+m)!]^{\frac{1}{2}}}{\Gamma(l+1-i\nu)}$$
$$\times \sum_{k=0}^{l-m} \frac{(-1)^k}{k!\,(l-m-k)!\,(l_0+m+k)!\,(l-l_0-k)!}\, G^{21}_{13}\left(\frac{p^2}{4}\middle|\begin{matrix} k+m+l_0-l \\ k+m+l_0-i\nu,\, m,\, l_0\end{matrix}\right). \tag{4.4}$$

We have then used the representation

$$P^l_{m,l_0}(c\theta) = (-1)^{l-l_0}\cdot[(l-l_0)!\,(l+l_0)!\,(l-m)!\,(l+m)!]^{\frac{1}{2}}$$
$$\times \sum_{k=0}^{l-m} \frac{(-1)^k}{k!\,(l-m-k)!\,(l_0+m+k)!\,(l-l_0-k)!}\left(c^2\frac{\theta}{2}\right)^{k+((m+l_0)/2)}\cdot\left(s^2\frac{\theta}{2}\right)^{l-k-((m+l_0)/2)}$$

for $P^l_{m,l_0}(c\theta)$ which follows immediately from (3.6).

In this example there is no splitting of the representation space, the change from the compact variable θ to the non-compact a is given by the same relation $a = \operatorname{tg}\theta/2$ in the whole interval of θ (excluding of course $\theta = \pi$). This is to be expected since there is only one double coset of $SL(2, C)$ with respect to K and the "spinor group" of $E(2)$. which has non-zero measure (cf. (2.4)). The present problem can also easily be treated with the global method of reference [8]. The starting point will then be the observation that in the decomposition

$$u_y(\theta) = z\cdot k$$

one has
$$z = \begin{pmatrix} 1 & 0 \\ \operatorname{tg}\theta/2 & 1\end{pmatrix}.$$

Since the derivations then follow exactly the lines given in [8] for the case of $SU(1,1)$, they are omitted here.

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Tryckt den 17 april 1967

Uppsala 1967. Almqvist & Wiksells Boktryckeri AB

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